Supplement for Online Publication to "The Fixed-b Limiting Distribution and the ERP of HAR Tests Under Nonstationarity"

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Abstract

This supplemental material is for online publication. It contains the proofs of the results.

S.A Mathematical Proofs

S.A.1 Proof of Theorem 3.1

Define

$$Q_T(r) = T^{-1} \sum_{t=1}^{\lfloor Tr \rfloor} x_t x_t', \qquad X_T(r) = T^{-1/2} S_{\lfloor Tr \rfloor}.$$

Let $K_b(\cdot) = K(\cdot/b)$ and

$$D_{b,T}(r) = T^2 \left[\left(K_b\left(\frac{\lfloor Tr \rfloor + 1}{T} \right) - K_b\left(\frac{\lfloor Tr \rfloor}{T} \right) \right) - \left(K_b\left(\frac{\lfloor Tr \rfloor}{T} \right) - K_b\left(\frac{\lfloor Tr \rfloor - 1}{T} \right) \right) \right].$$

By symmetry of $K(\cdot)$, it follows the symmetry of $D_{b,T}(\cdot)$. If $K''(r)$ is assumed to exist, then $\lim_{T\to\infty} D_{b,T}(r) = K_b''(r)$. The convergence is uniform in r if $K''(r)$ is continuous. From Assumption 2.1-2.2 it follows that $(Q_T(r), X_T(r)', D_{b,T}(r)) \Rightarrow (\int_0^r Q(u)\,du, (\int_0^r \Sigma(u)\,dW_p(u))', K_b''(r))$ jointly.

Define $K_{i,j} = ((i-j)/(bT))$. We have

$$\widehat{\Omega}_{\text{fixed}-b} = T^{-1} \sum_{i=1}^{T} \sum_{j=1}^{T} K_{i,j} \widehat{V}_i \widehat{V}_j' = T^{-1} \sum_{i=1}^{T} \widehat{V}_i \left(\sum_{j=1}^{T} K_{i,j} \widehat{V}_j' \right).$$

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Note that

$$\begin{aligned} & T^2\left(\left(K_{i,j}-K_{i,j+1}\right)-\left(K_{i+1,j}-K_{i+1,j+1}\right)\right) \\ & =-T^2\left[\left(K_b\left(\frac{i-j+1}{T}\right)-K_b\left(\frac{i-j}{T}\right)\right)-\left(K_b\left(\frac{i-j}{T}\right)-K_b\left(\frac{i-j-1}{T}\right)\right)\right] \\ & =D_{b,T}\left(\left(i-j\right)/T\right) \end{aligned} \tag{S.1}$$

Define $\widehat{S}_t=\sum_{j=1}^{t}\widehat{V}_j$. Note that $\widehat{S}_T=\mathbf{0}$ by the normal equations for OLS. We have

$$\begin{aligned} T^{-1/2}\widehat{S}_{\lfloor Tr\rfloor} & =T^{-1/2}S_{\lfloor Tr\rfloor}-T^{-1}\sum_{t=1}^{\lfloor Tr\rfloor}x_t x_t'\left(T^{-1}\sum_{t=1}^{T}x_t x_t'\right)^{-1}T^{-1/2}S_T \\ & =X_T\left(r\right)-Q_T\left(r\right)Q_T\left(1\right)^{-1}X_T\left(1\right). \end{aligned} \tag{S.2}$$

Using the identity

$$\sum_{l=1}^{T}a_l b_l=\sum_{l=1}^{T-1}\left(\left(a_l-a_{l+1}\right)\sum_{j=1}^{l}b_j\right)+a_T\sum_{j=1}^{T}b_j, \tag{S.3}$$

first applied to $\sum_{j=1}^{T}K_{i,j}\widehat{V}_j'$ and then again to the sum over i, Kiefer and Vogelsang (2002) showed that

$$\widehat{\Omega}_{\text{fixed}-b}=T^{-1}\sum_{i=1}^{T-1}T^{-1}\sum_{j=1}^{T-1}T^2\left(\left(K_{i,j}-K_{i,j+1}\right)-\left(K_{i+1,j}-K_{i+1,j+1}\right)\right)T^{-1/2}\widehat{S}_i T^{-1/2}\widehat{S}_j'. \tag{S.4}$$

We first consider part (i). Using (S.1)-(S.2) in (S.4) we have

$$\begin{aligned} \widehat{\Omega}_{\text{fixed}-b} & =\int_0^1\int_0^1-D_{b,T}\left(r-s\right)\left[X_T\left(r\right)-Q_T\left(r\right)Q_T\left(1\right)^{-1}X_T\left(1\right)\right]\left[X_T\left(s\right)-Q_T\left(s\right)Q_T\left(1\right)^{-1}X_T\left(1\right)\right]'drds \\ & \Rightarrow-\int_0^1\int_0^1K_b''\left(r-s\right)\left(\int_0^r\Sigma\left(u\right)dW_p\left(u\right)-\left(\int_0^r Q\left(u\right)du\right)\overline{Q}^{-1}\int_0^1\Sigma\left(u\right)dW_p\left(u\right)\right) \\ & \quad\times\left(\int_0^s\Sigma\left(u\right)dW_p\left(u\right)-\left(\int_0^s Q\left(u\right)du\right)\overline{Q}^{-1}\int_0^1\Sigma\left(u\right)dW_p\left(u\right)\right)'drds \\ & =-\frac{1}{b^2}\int_0^1\int_0^1K''\left(\frac{r-s}{b}\right)\widetilde{B}_p\left(r,\,\Sigma,\,Q\right)\widetilde{B}_p\left(s,\,\Sigma,\,Q\right)'drds \\ & =\mathscr{G}_b, \end{aligned}$$

where we have used Assumption 2.1-2.2, the continuous mapping theorem since $\widehat{\Omega}_{\text{fixed}-b}$ is a continuous function of $\left(Q_T\left(r\right),\,X_T\left(r\right)',\,D_{b.T}\left(r\right)\right)$ and $K_b''\left(x\right)=b^{-2}K''\left(x/b\right)$.

We now move to part (ii). Suppose that the Bartlett kernel K_{BT} is used. Let

$$\Delta^2K_{ij}\triangleq\left(K_{i,j}-K_{i,j+1}\right)-\left(K_{i+1,j}-K_{i+1,j+1}\right).$$

Note that $\Delta^2K_{i,j}=2/(bT)$ for $|i-j|=0$, $\Delta^2K_{i,j}=-1/\left(bT\right)+1-\lfloor bT\rfloor/\left(bT\right)$ for $|i-j|=\lfloor bT\rfloor$,

$\Delta^2 K_{i,j} = -(1 - \lfloor bT \rfloor / bT)$ for $|i-j| = \lfloor bT \rfloor + 1$ and $\Delta^2 K_{i,j} = 0$ otherwise. Using this into (S.4) we obtain

$$\begin{aligned}
\widehat{\Omega}_{\text{fixed}-b} &= T^{-1} \sum_{i=1}^{T-1} T^{-1} \sum_{j=1}^{T-1} T^2 \Delta^2 K_{i,j} T^{-1/2} \widehat{S}_i T^{-1/2} \widehat{S}_j' \\
&= \frac{2}{bT} \sum_{i=1}^{T-1} T^{-1/2} \widehat{S}_i T^{-1/2} \widehat{S}_i' \\
&\quad + T \left[-\frac{1}{bT} + 1 - \frac{\lfloor bT \rfloor}{bT} \right] T^{-1} \sum_{i=1}^{T-\lfloor bT \rfloor - 1} \left(T^{-1/2} \widehat{S}_{i+\lfloor bT \rfloor} T^{-1/2} \widehat{S}_i' + T^{-1/2} \widehat{S}_i T^{-1/2} \widehat{S}_{i+\lfloor bT \rfloor}' \right) \\
&\quad - \left(1 - \frac{\lfloor bT \rfloor}{bT} \right) \sum_{i=1}^{T-[bT]-2} \left(T^{-1/2} \widehat{S}_{i+\lfloor bT \rfloor + 1} T^{-1/2} \widehat{S}_i' + T^{-1/2} \widehat{S}_i T^{-1/2} \widehat{S}_{i+\lfloor bT \rfloor + 1}' \right) \\
&= \frac{2}{bT} \sum_{i=1}^{T-1} T^{-1/2} \widehat{S}_i T^{-1/2} \widehat{S}_i' \\
&\quad + T \left[-\frac{1}{bT} + 1 - \frac{\lfloor bT \rfloor}{bT} \right] T^{-1} \sum_{i=1}^{T-\lfloor bT \rfloor - 1} \left(T^{-1/2} \widehat{S}_{i+\lfloor bT \rfloor} T^{-1/2} \widehat{S}_i' + T^{-1/2} \widehat{S}_i T^{-1/2} \widehat{S}_{i+\lfloor bT \rfloor}' \right) \\
&\quad - \left(1 - \frac{\lfloor bT \rfloor}{bT} \right) \sum_{i=1}^{T-[bT]-1} \left(T^{-1/2} \widehat{S}_{i+\lfloor bT \rfloor + 1} T^{-1/2} \widehat{S}_i' + T^{-1/2} \widehat{S}_i T^{-1/2} \widehat{S}_{i+\lfloor bT \rfloor + 1}' \right),
\end{aligned}$$

where we have used that $\widehat{S}_T \widehat{S}_{T-\lfloor bT \rfloor - 1}' = \mathbf{0}$ and $\widehat{S}_{T-\lfloor bT \rfloor - 1} \widehat{S}_T' = \mathbf{0}$. Note that

$$\begin{aligned}
&\left(1 - \frac{\lfloor bT \rfloor}{bT} \right) \sum_{i=1}^{T-\lfloor bT \rfloor - 1} T^{-1/2} \widehat{S}_{i+[bT]+1} T^{-1/2} \widehat{S}_i' \\
&\quad = \left(1 - \frac{\lfloor bT \rfloor}{bT} \right) \sum_{i=1}^{T-\lfloor bT \rfloor - 1} \left(T^{-1/2} \widehat{S}_{i+\lfloor bT \rfloor} T^{-1/2} \widehat{S}_i' + T^{-1/2} \widehat{V}_{i+\lfloor bT \rfloor + 1} T^{-1/2} \widehat{S}_i' \right) \\
&\quad = \left(1 - \frac{\lfloor bT \rfloor}{bT} \right) \sum_{i=1}^{T-\lfloor bT \rfloor - 1} T^{-1/2} \widehat{S}_{i+\lfloor bT \rfloor} T^{-1/2} \widehat{S}_i' + \left(1 - \frac{\lfloor bT \rfloor}{bT} \right) T^{-1} \sum_{i=1}^{T-\lfloor bT \rfloor - 1} \widehat{V}_{i+\lfloor bT \rfloor + 1} \widehat{S}_i' \\
&\quad = \left(1 - \frac{\lfloor bT \rfloor}{bT} \right) \sum_{i=1}^{T-\lfloor bT \rfloor - 1} T^{-1/2} \widehat{S}_{i+\lfloor bT \rfloor} T^{-1/2} \widehat{S}_i' + o_{\mathbb{P}}(1),
\end{aligned}$$

where the $o_{\mathbb{P}}(1)$ term follows from $\lim_{T\to\infty} \left(1 - \frac{\lfloor bT \rfloor}{bT} \right) = 0$ and $T^{-1} \sum_{i=1}^{T-\lfloor bT \rfloor - 1} \widehat{V}_{i+\lfloor bT \rfloor + 1} \widehat{S}_i' = O_{\mathbb{P}}(1)$. It follows that

$$\begin{aligned}
\widehat{\Omega}_{\text{fixed}-b} &= \frac{2}{bT} \sum_{i=1}^{T-1} T^{-1/2} \widehat{S}_i T^{-1/2} \widehat{S}_i' \\
&\quad - \frac{1}{bT} \sum_{i=1}^{T-\lfloor bT \rfloor - 1} \left(T^{-1/2} \widehat{S}_{i+\lfloor bT \rfloor} T^{-1/2} \widehat{S}_i' + T^{-1/2} \widehat{S}_i T^{-1/2} \widehat{S}_{i+\lfloor bT \rfloor}' \right) + o_{\mathbb{P}}(1).
\end{aligned}$$

Using (S.2) and Assumption 2.1-2.2 we yield,

$$\begin{aligned}
\widehat{\Omega}_{\text{fixed}-b} \Rightarrow & \frac{2}{b}\int_0^1 \left(\int_0^r \Sigma(u)\,dW_p(u) - \left(\int_0^r Q(u)\,du\right)\overline{Q}^{-1}\int_0^1 \Sigma(u)\,dW_p(u)\right) \\
& \times \left(\int_0^r \Sigma(u)\,dW_p(u) - \left(\int_0^r Q(u)\,du\right)\overline{Q}^{-1}\int_0^1 \Sigma(u)\,dW_p(u)\right)' dr \\
& - \frac{1}{b}\int_0^{1-b} \left(\left(\int_0^{r+b} \Sigma(u)\,dW_p(u) - \left(\int_0^{r+b} Q(u)\,du\right)\overline{Q}^{-1}\int_0^1 \Sigma(u)\,dW_p(u)\right)\right. \\
& \times \left(\int_0^r \Sigma(u)\,dW_p(u) - \left(\int_0^r Q(u)\,du\right)\overline{Q}^{-1}\int_0^1 \Sigma(u)\,dW_p(u)\right)' \\
& + \left(\int_0^r \Sigma(u)\,dW_p(u) - \left(\int_0^r Q(u)\,du\right)\overline{Q}^{-1}\int_0^1 \Sigma(u)\,dW_p(u)\right) \\
& \times \left.\left(\int_0^{r+b} \Sigma(u)\,dW_p(u) - \left(\int_0^{r+b} Q(u)\,du\right)\overline{Q}^{-1}\int_0^1 \Sigma(u)\,dW_p(u)\right)'\right) dr \\
= & \frac{2}{b}\int_0^1 \widetilde{B}_p(r,\Sigma,Q)\,\widetilde{B}_p(r,\Sigma,Q)'\,dr \\
& - \frac{1}{b}\int_0^{1-b}\left(\widetilde{B}_p(r+b,\Sigma,Q)\,\widetilde{B}_p(r,\Sigma,Q)' + \widetilde{B}_p(r,\Sigma,Q)\,\widetilde{B}_p(r+b,\Sigma,Q)'\right) dr,
\end{aligned}$$

which concludes the proof. □

S.A.2 Proof of Theorem 3.2

We begin with part (i). Using Theorem 3.1 we have

$$\begin{aligned}
F_{\text{fixed}-b} = & \left(RQ_T(1)^{-1}X_T(1)\right)'\left(RQ_T(1)^{-1}\widehat{\Omega}_{\text{fixed}-b}Q_T(1)^{-1}R'\right)^{-1}RQ_T(1)^{-1}X_T(1)/q \\
\Rightarrow & \left(R\overline{Q}^{-1}\int_0^1 \Sigma(u)\,dW_p(u)\right)'\left(R\overline{Q}^{-1}\mathscr{G}_b\overline{Q}^{-1}R'\right)^{-1}R\overline{Q}^{-1}\int_0^1 \Sigma(u)\,dW_p(u)/q
\end{aligned}$$

where $\mathscr{G}_b$ is defined in (3.3). If $q=1$, then

$$\begin{aligned}
t_{\text{fixed}-b} = & \frac{RQ_T(1)^{-1}X_T(1)}{\sqrt{RQ_T(1)^{-1}\widehat{\Omega}_{\text{fixed}-b}Q_T(1)^{-1}R'}} \\
\Rightarrow & \frac{R\overline{Q}^{-1}\int_0^1 \Sigma(u)\,dW_p(u)}{\sqrt{R\overline{Q}^{-1}\mathscr{G}_b\overline{Q}^{-1}R'}}.
\end{aligned}$$

Part (ii) can be proved in a similar manner. □

S.A.3 Proof of Theorem 4.1

Throughout the proof, let $\widehat{\Omega}_b = \widehat{\Omega}_{\text{fixed}-b}$ and

$$Z_{T,0}(z) \triangleq \mathbb{P}\left(\left|\frac{\sqrt{T}\left(\widehat{\beta}-\beta_0\right)}{\sqrt{\widehat{\Omega}_b}}\right| \leq z\right), \qquad Z_0(z) \triangleq \mathbb{P}\left(\left|\frac{\int_0^1 \Sigma(u)\, dW_1(u)}{\sqrt{\mathscr{G}_b}}\right| \leq z\right),$$

$$\widehat{Z}_0(z) \triangleq \mathbb{P}\left(\left|\frac{\int_0^1 \widehat{\Sigma}(u)\, dW_1(u)}{\sqrt{\widehat{\mathscr{G}}_b}}\right| \leq z\right).$$

We have

$$\begin{aligned}\sup_{z\in\mathbb{R}_+}\left|Z_{T,0}(z)-\widehat{Z}_0(z)\right| &\leq \sup_{z\in\mathbb{R}_+}\left|Z_{T,0}(z)-Z_0(z)\right| + \sup_{z\in\mathbb{R}_+}\left|Z_0(z)-\widehat{Z}_0(z)\right| \\ &\triangleq D_1 + D_2.\end{aligned}$$

We show that $D_1 = O(T^{-1})$ and $D_2 = O((Th_1h_2)^{-1/2})$. We begin with some preliminary lemmas. The first lemma below generalizes Theorem 3.1 to allow for general kernels satisfying Assumption 4.2 and for a p-vector $\widehat{V}_t$. Let

$$\widetilde{B}_p(r) = \int_0^r \Sigma(u)\, dW_p(u) - r\left(\int_0^1 \Sigma(u)\, dW_p(u)\right),$$

and

$$K_b^*(r, s) \triangleq K_b(r-s) - \int_0^1 K_b(r-t)\, dt - \int_0^1 K_b(\tau - s)\, d\tau + \int_0^1\int_0^1 K_b(t-\tau)\, dt d\tau.$$

Lemma S.A.1. *Let Assumption 2.1 and 4.2 hold. We have:*
(i) $\widehat{\Omega}_b \Rightarrow \mathscr{G}_b$ *where* $\mathscr{G}_b = \int_0^1\int_0^1 K_b(r-s)\, d\widetilde{B}_p(r)\, d\widetilde{B}_p(s)'$;
(ii) $\mu_b = \mathbb{E}(\mathscr{G}_b) = \int_0^1 K_b^*(s, s)\, \Omega(s)\, ds$.

Proof of Lemma S.A.1. We begin with part (i). Since $K(\cdot)$ is positive semidefinite, Mercer's Theorem implies that

$$K(r-s) = \sum_{n=1}^{\infty} \frac{1}{\lambda_n} g_n(r)\, g_n(s), \tag{S.5}$$

where $\lambda_n^{-1} > 0$ are the eigenvalues of $K(\cdot)$ and $g_n(\cdot)$ are the corresponding eigenfunctions, i.e., $g_n(s) = \lambda_n \int_0^1 K(r-s)\, g_n(r)\, dr$. The convergence of the right-hand side over $(r, s) \in [0, 1] \times [0, 1]$ is uniform.

Using Assumption 2.1 and (S.5) we have

$$\begin{aligned}\widehat{\Omega}_b &= \frac{1}{T}\sum_{t=1}^{T}\sum_{s=1}^{T} K_b\left(\frac{t-s}{T}\right)\widehat{V}_t\widehat{V}_s' \\ &= \sum_{n=1}^{\infty}\frac{1}{\lambda_n}\frac{1}{T}\sum_{t=1}^{T}\sum_{s=1}^{T} g_n(t/(bT))\, g_n(s/(bT))\, \widehat{V}_t\widehat{V}_s'\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \left(\frac{1}{\sqrt{T}} \sum_{t=1}^{T} \widehat{V}_t g_n \left(t/(bT)\right) \right) \left(\frac{1}{\sqrt{T}} \sum_{s=1}^{T} g_n \left(s/(bT)\right) \widehat{V}_s' \right) \\
&\Rightarrow \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \left(\int_0^1 g_n \left(r/b\right) d\widetilde{B}_p \left(r\right) \right) \left(\int_0^1 g_n \left(s/b\right) d\widetilde{B}_p \left(s\right) \right)' \\
&= \int_0^1 \int_0^1 \sum_{n=1}^{\infty} \frac{1}{\lambda_n} g_n \left(r/b\right) g_n \left(s/b\right) d\widetilde{B}_p \left(r\right) d\widetilde{B}_p \left(s\right)' \\
&= \int_0^1 \int_0^1 K_b \left(r-s\right) d\widetilde{B}_p \left(r\right) d\widetilde{B}_p \left(s\right)' . \\
&= \mathscr{G}_b
\end{aligned}
$$

For part (ii), after some algebraic manipulations we can write

$$
\mathscr{G}_b = \int_0^1 \int_0^1 K_b^* \left(r,\, s\right) \Sigma \left(r\right) dW_p \left(r\right) \left(\Sigma \left(s\right) dW_p \left(s\right)\right)' .
$$

It follows that

$$
\begin{aligned}
\mathbb{E} \left(\mathscr{G}_b\right) &= \mathbb{E} \left(\int_0^1 \int_0^1 K_b^* \left(r,\, s\right) \Sigma \left(r\right) dW_p \left(r\right) \left(\Sigma \left(s\right) dW_p \left(s\right)\right)' \right) \\
&= \int_0^1 K_b^* \left(s,\, s\right) \Sigma \left(s\right) \Sigma \left(s\right)' ds \\
&= \int_0^1 K_b^* \left(s,\, s\right) \Omega \left(s\right) ds,
\end{aligned}
$$

which concludes the proof. □

Let $p = 1$. Sun, Phillips and Jin (2008) showed that $K_b^* \left(r,\, s\right)$ is positive semidefinite. Thus, using Mercer's theorem, we have

$$
K_b^* \left(r,\, s\right) = \sum_{n=1}^{\infty} \lambda_n^* g_n^* \left(r\right) g_n^* \left(s\right), \tag{S.6}
$$

where $\lambda_n^* > 0$ are the eigenvalues of $K_b^* \left(\cdot,\, \cdot\right)$ and $g_n^* \left(r\right)$ are the corresponding eigenfunctions, i.e., $\lambda_n^* g_n^* \left(s\right) = \int_0^\infty K_b^* \left(r,\, s\right) g_n^* \left(r\right) dr$. Since $\Sigma \left(s\right) > 0$, we can write

$$
K_b^* \left(r,\, s\right) \Sigma \left(r\right) \Sigma \left(s\right) = \sum_{n=1}^{\infty} \lambda_n^* g_n^{**} \left(r\right) g_n^{**} \left(s\right),
$$

where $g_n^{**} \left(r\right) = \Sigma \left(r\right) g_n^* \left(r\right)$. Then, $\lambda_n^* g_n^{**} \left(s\right) = \int_0^\infty K_b^* \left(r,\, s\right) \Sigma \left(r\right) \Sigma \left(s\right) g_n^{**} \left(r\right) dr$. It follows that $\mathscr{G}_b = \sum_{n=1}^{\infty} \lambda_n^* Z_n^2$ where $Z_n \sim i.i.d. \mathscr{N}(0,\, 1)$. Thus, the characteristic function of $\Omega^{-1}(\mathscr{G}_b - \mu_b)$ is given by

$$
\psi \left(t\right) = \mathbb{E} \left(e^{it\Omega^{-1}(\mathscr{G}_b - \mu_b)} \right) = \prod_{n=1}^{\infty} \left(1 - 2i\lambda_n^* \Omega^{-1} t\right)^{-1/2} \left(e^{-it\Omega^{-1}\mu_b} \right), \tag{S.7}
$$

and the cumulant generating function is

$$\log \psi(t) = \sum_{m=2}^{\infty} \left(2^{m-1}(m-1)! \sum_{n=1}^{\infty} \left(\lambda_n^* \Omega^{-1} \right)^m \right) \frac{(it)^m}{m!}. \tag{S.8}$$

Let κ_j $(j = 1, \ldots)$ be the jth cumulant of $\Omega^{-1}(\mathscr{G}_b - \mu_b)$. Then

$$\kappa_1 = 0 \quad \text{and} \quad \kappa_m = 2^{m-1}(m-1)! \sum_{n=1}^{\infty} \left(\lambda_n^* \Omega^{-1} \right)^m \qquad \text{for } m \geq 2. \tag{S.9}$$

Let $\tau_{m+1} = \tau_1$. For $m \geq 2$, some algebraic manipulations show that

$$\begin{aligned} \kappa_m &= 2^{m-1}(m-1)!\Omega^{-m} \sum_{n=1}^{\infty} \left(\left(g_n^{**}(s) \right)^{-1} \int_0^{\infty} K_b^*(r, s) \Sigma(r) \Sigma(s) g_n^{**}(r)\, dr \right)^m \\ &= 2^{m-1}(m-1)!\Omega^{-m} \int_0^1 \cdots \int_0^1 \left(\prod_{j=1}^{m} \Omega(\tau_j) K_b^*(\tau_j, \tau_{j+1}) \right) d\tau_1 \cdots d\tau_m. \end{aligned} \tag{S.10}$$

Let $\Xi_m = \Omega^{-m}\mathbb{E}((\mathscr{G}_b - \mu_b)^m)$ for $m \geq 1$.

Lemma S.A.2. *Let* $\overline{C}_1 = 4\int_{-\infty}^{\infty} |K(v)|dv$, $C_\Omega = \sup_{s\in[0,1]} \Omega(s)$, $D_m > 0$ *be a constant depending on* m *and Assumption 4.2 hold. Then, for* $m \geq 1$, *we have*

$$|\kappa_m| \leq 2^m (m-1)!\Omega^{-m} C_\Omega^m \left(\overline{C}_1 b \right)^{m-1}, \tag{S.11}$$

and

$$|\Xi_m| \leq D_m 2^{2m} m! \Omega^{-m} C_\Omega^m \left(\overline{C}_1 b \right)^{m-1}. \tag{S.12}$$

Proof of Lemma S.A.2. From eq. (A.3) in Sun et al. (2008),[1]

$$\left| \int_0^1 \cdots \int_0^1 \left(\prod_{j=1}^{m} K_b^*(\tau_j, \tau_{j+1}) \right) d\tau_1 \cdots d\tau_m \right| \leq 2 \left(\sup_s \int_0^1 |K_b^*(r, s)|\, dr \right)^{m-1}. \tag{S.13}$$

We have

$$\sup_s \int_0^1 |K_b^*(r, s)|\, dr \leq b\overline{C}_1, \tag{S.14}$$

from which it follows that

$$\left| \int_0^1 \cdots \int_0^1 \left(\prod_{j=1}^{m} K_b^*(\tau_j, \tau_{j+1}) \right) d\tau_1 \cdots d\tau_m \right| \leq 2 \left(b\overline{C}_1 \right)^{m-1}. \tag{S.15}$$

[1] Any reference to equations in Sun et al. (2008) corresponds to the long version of the working paper available in Sun's webpage.

Using (S.15) and some algebraic manipulations we have for $m \geq 2$,

$$\begin{aligned}\kappa_m &= 2^{m-1}(m-1)!\Omega^{-m}\sum_{n=1}^{\infty}(\lambda_n^*)^m \\ &= 2^{m-1}(m-1)!\Omega^{-m}\int_0^1\cdots\int_0^1\left(\prod_{j=1}^{m}\Omega(\tau_j)K_b^*(\tau_j,\,\tau_{j+1})\right)d\tau_1\cdots d\tau_m \\ &\leq 2^{m-1}(m-1)!\Omega^{-m}C_\Omega^m\int_0^1\cdots\int_0^1\left|\prod_{j=1}^{m}K_b^*(\tau_j,\,\tau_{j+1})\right|d\tau_1\cdots d\tau_m \\ &\leq 2^m(m-1)!\Omega^{-m}C_\Omega^m\left(b\overline{C}_1\right)^{m-1},\end{aligned} \tag{S.16}$$

where $C_\Omega = \sup_{s\in[0,\,1]}\Omega(s)$. The moments $\{\Xi_j\}$ and cumulants $\{\kappa_j\}$ are related by the following

$$\Xi_m = \sum_{\pi_p}\frac{m!}{(j_1!)^{m_1}(j_2!)^{m_2}\cdots(j_l!)^{m_l}}\frac{1}{m_1!m_2!\cdots m_l!}\prod_{j=\pi_p}\kappa_j, \tag{S.17}$$

where the sum is taken over all partitions $\pi_p \in \Pi$ such that

$$\pi_p = \left[\underbrace{j_1,\cdots j_1}_{m_1\text{ times}},\,\underbrace{j_2,\ldots,\,j_2}_{m_2\text{ times}},\ldots,\,\underbrace{j_l,\ldots,\,j_l}_{m_l\text{ times}}\right], \tag{S.18}$$

for some integer l, sequence $\{j_i\}_{i=1}^{l}$ such that $j_1 > j_2 > \cdots > j_l$ and $m = \sum_{i=1}^{l} m_i j_i$.

Using (S.16)-(S.18) yield

$$\begin{aligned}|\Xi_m| &< 2^m m!\Omega^{-m}C_\Omega^m\left(\overline{C}_1 b\right)^{m-1}\sum_{\pi}\frac{(j_1)^{-m_1}(j_2)^{-m_2}\cdots(j_l)^{-m_l}}{m_1!m_2!\cdots m_l!}j_1^m \\ &\leq D_m 2^{2m} m!\Omega^{-m}C_\Omega^m\left(\overline{C}_1 b\right)^{m-1},\end{aligned} \tag{S.19}$$

where the last inequality follows from

$$\sum_{\pi_p}\frac{(j_1)^{-m_1}(j_2)^{-m_2}\cdots(j_l)^{-m_l}}{m_1!m_2!\cdots m_l!} \leq \sum_{\pi_p}\frac{1}{m_1!m_2!\cdots m_l!} < 2^m, \tag{S.20}$$

and $D_m = \sup_{\pi_p\in\Pi}(j_1 \in \pi_p)$. □

We now develop an asymptotic expansion of $Z_{T,0} = \mathbb{P}(\sqrt{T}(\widehat{\beta}-\beta_0)/\sqrt{\widehat{\Omega}_b} \leq z)$ for $\beta = \beta_0 + d/\sqrt{T}$. When $d = 0$ (resp., $d \neq 0$) the expansion can be used to approximate the size (resp., power) of the t-statistic. Since V_t is autocorrelated, $\widehat{\beta}$ and $\widehat{\Omega}_b$ are statistically dependent. Thus, we decompose $\widehat{\beta}$ and $\widehat{\Omega}_b$ into statistically independent components. Let $V = (V_1,\ldots,\,V_T)'$, $y = (y_1,\ldots,\,y_T)'$, $l_T = (1,\ldots,\,1)'$ and $\Upsilon_T = \mathrm{Var}(V)$. The GLS estimator of β is $\widetilde{\beta} = (l_T'\Upsilon_T^{-1}l_T)^{-1}l_T'\Upsilon_T^{-1}y$. Then,

$$\widehat{\beta} - \beta = \widetilde{\beta} - \beta + \left(l_T'l_T\right)^{-1}l_T'\widetilde{V}, \tag{S.21}$$

where $\widetilde{V} = (I - l_T(l_T'\Upsilon_T^{-1}l_T)^{-1}l_T'\Upsilon_T^{-1})V$, which is statistically independent of $\widetilde{\beta}-\beta$. Since $\widehat{\Omega}_b$ can be written as a quadratic form in $\widetilde{V}$, it is also statistically independent of $\widetilde{\beta}-\beta$. From Casini (2023)

$$\Omega_T \triangleq \mathrm{Var}\left(\sqrt{T}\left(\widehat{\beta}-\beta\right)\right) = T^{-1}l_T'\Upsilon_T l_T = \Omega + O\left(T^{-1}\right), \tag{S.22}$$

where $\Omega = 2\pi\int_0^1 f(u,\,0)\,du$. Similarly, one can show that

$$\widetilde{\Omega}_T \triangleq \mathrm{Var}\left(\sqrt{T}\left(\widetilde{\beta}-\beta\right)\right) = T\left(l_T'\Upsilon_T^{-1}l_T\right)^{-1} = \Omega + O\left(T^{-1}\right). \tag{S.23}$$

Therefore $T^{-1/2}l_T'\widetilde{V} = \mathscr{N}(0,\,O(T^{-1}))$. As in eq. (45) in Sun et al. (2008), using the independence of $\widetilde{\beta}$ and $(\widetilde{V},\,\widehat{\Omega}_b)$, we have

$$\begin{aligned}&\mathbb{P}\left(\sqrt{T}\left(\widehat{\beta}-\beta_0\right)/\sqrt{\widehat{\Omega}_b}\le z\right) \\ &\quad= \mathbb{P}\left(\sqrt{T}\left(\widetilde{\beta}-\beta\right)/\sqrt{\widetilde{\Omega}_T}+d/\sqrt{\widetilde{\Omega}_T}\le z\sqrt{\widehat{\Omega}_b/\widetilde{\Omega}_T}\right)+O\left(T^{-1}\right),\end{aligned} \tag{S.24}$$

uniformly over $z\in\mathbb{R}$ where Φ and φ are the cdf and pdf of the standard normal distribution, respectively.

Similarly, uniformly over $z\in\mathbb{R}$, we have

$$\mathbb{P}\left(\sqrt{T}\left(\widehat{\beta}-\beta_0\right)/\sqrt{\widehat{\Omega}_b}\le -z\right) = \mathbb{P}\left(\sqrt{T}\left(\widetilde{\beta}-\beta\right)/\sqrt{\widetilde{\Omega}_T}+c/\sqrt{\widetilde{\Omega}_T}\le -z\sqrt{\widehat{\Omega}_b/\widetilde{\Omega}_T}\right)+O\left(T^{-1}\right).$$

It follows that

$$\begin{aligned}Z_{T,d}(z) &= \mathbb{P}\left(\left|\sqrt{T}\left(\widehat{\beta}-\beta_0\right)/\sqrt{\widehat{\Omega}_b}\right|\le z\right) \\ &= \mathbb{P}\left(\left(\sqrt{T}\left(\widetilde{\beta}-\beta\right)/\sqrt{\widetilde{\Omega}_T}+d/\sqrt{\widetilde{\Omega}_T}\right)^2\le z^2\widehat{\Omega}_b/\widetilde{\Omega}_T\right)+O\left(T^{-1}\right) \\ &= \mathbb{E}\left(G_{\widetilde{d}}\left(z^2\widehat{\Omega}_b/\widetilde{\Omega}_T\right)\right) = \mathbb{E}\left(G_{\widetilde{d}}\left(z^2\zeta_{b,T}\right)\right)+O\left(T^{-1}\right),\end{aligned} \tag{S.25}$$

uniformly over $z\in\mathbb{R}_+$, where $G_{\widetilde{d}}(z) = G(z;\,\widetilde{d})$ is the cdf of a non-central chi-squared $\chi_1(\widetilde{d}^2)$ with noncentrality parameter $\widetilde{d}^2 = d^2/\Omega_T$ and $\zeta_{b,T} = \widehat{\Omega}_b/\Omega_T$. Note that $\zeta_{b,T}\Rightarrow\mathscr{G}_b/\Omega$. Let $\mu_{b,T} = \mathbb{E}(\zeta_{b,T})$ and consider a fourth-order Taylor expansion of $\zeta_{b,T}$ around its mean,

$$\begin{aligned}Z_{T,d}(z) &= G_{\widetilde{d}}\left(\mu_{b,T}z^2\right)+\frac{1}{2}G''_{\widetilde{d}}\left(\mu_{b,T}z^2\right)\mathbb{E}\left(\zeta_{b,T}-\mu_{b,T}\right)^2z^4 \\ &\quad+\frac{1}{6}G'''_{\widetilde{d}}\left(\mu_{b,T}z^2\right)\mathbb{E}\left(\zeta_{b,T}-\mu_{b,T}\right)^3z^6+O\left(\mathbb{E}\left(\zeta_{b,T}-\mu_{b,T}\right)^4\right)+O\left(T^{-1}\right),\end{aligned} \tag{S.26}$$

where the $O(\cdot)$ term holds uniformly over $z\in\mathbb{R}_+$.

Using (S.25) we have

$$Z_{T,0}(z)-Z_0(z) = \mathbb{P}\left(\left|\frac{\sqrt{T}\left(\widehat{\beta}-\beta_0\right)}{\sqrt{\widehat{\Omega}_b}}\right|\le z\right)-\mathbb{P}\left(\left|\frac{\int_0^1\Sigma(u)\,dW_1(u)}{\sqrt{\mathscr{G}_b}}\right|\le z\right) \tag{S.27}$$

$$= \mathbb{E}\left(F_{\chi}\left(z^2\zeta_{b,T}\right)\right) - \mathbb{E}\left(F_{\chi}\left(z^2\mathscr{G}_b/\Omega\right)\right) + O\left(T^{-1}\right),$$

where $F_{\chi}(\cdot) = G_0(\cdot)$ is the cdf of the χ_1^2 distribution. Next, we compute the cumulants of both $\zeta_{b,T} - \mu_{b,T}$ and $\Omega^{-1}(\mathscr{G}_b - \mu_b)$ where $\mu_b = \mathbb{E}(\mathscr{G}_b)$. Note that $\zeta_{b,T}$ is a quadratic form in a Gaussian vector since $\widehat{\Omega}_b = T^{-1}\widehat{V}'W_b\widehat{V} = T^{-1}V'A_TW_bA_TV$, where W_b is $T \times T$ with $(j,\, s)$-th element $K_b((j-s)/T)$ and $A_T = I_T - l_Tl_T'/T$. The characteristic function of $\zeta_{b,T} - \mu_{b,T}$ is given by

$$\psi_{b,T}(t) = \left| I - 2it\frac{\Upsilon_T A_T W_b A_T}{T\Omega_T}\right|^{-1/2} \exp\left(-it\mu_{b,T}\right), \tag{S.28}$$

where $\Upsilon_T = \mathbb{E}(uu')$ and the cumulant generating function is

$$\log\left(\psi_{b,T}(t)\right) = -\frac{1}{2}\log\left| I - 2it\frac{\Upsilon_T A_T W_b A_T}{T\Omega_T}\right| - it\mu_{b,T} = \sum_{m=1}^{\infty}\kappa_{m,T}\frac{(it)^m}{m!}, \tag{S.29}$$

where $\kappa_{m,T}$ is the mth cumulant of $\zeta_{b,T} - \mu_{b,T}$. Note that $\kappa_{1,T} = 0$ and

$$\kappa_{m,T} = 2^{m-1}(m-1)!T^{-m}(\Omega_T)^{-m}\,\mathrm{Tr}\left(\left(\Upsilon_T A_T W_b A_T\right)^m\right), \qquad m \geq 2. \tag{S.30}$$

Lemma S.A.3. *Let Assumption 4.1-4.2 hold. We have: (i)* $\mu_{b,T} = \Omega^{-1}\mu_b + O\left(T^{-1}\right)$; *(ii)* $\kappa_{m,T} = \kappa_m + O(m!2^mT^{-2}(\overline{C}_1b)^{m-2})$ *uniformly over* $m \geq 1$; *(iii)* $\Xi_{m,T} = \mathbb{E}(\zeta_{b,T} - \mu_{b,T})^m = \Xi_m + O(m!2^{2m}T^{-2}(\overline{C}_1b)^{m-2})$.

Proof of Lemma S.A.3. Note that $\mu_{b,T} = (T\Omega_T)^{-1}\mathrm{Tr}(\Upsilon_T A_T W_b A_T)$. Let $\widetilde{W}_b = A_T W_b A_T$, where its $(j,\, s)$-th element is given by

$$\widetilde{K}_b\left(\frac{j}{T},\,\frac{s}{T}\right) = K_b\left(\frac{j-s}{T}\right) - \frac{1}{T}\sum_{p=1}^{T}K_b\left(\frac{j-p}{T}\right) - \frac{1}{T}\sum_{q=1}^{T}K_b\left(\frac{q-s}{T}\right) + \frac{1}{T^2}\sum_{p=1}^{T}\sum_{q=1}^{T}K_b\left(\frac{p-q}{T}\right). \tag{S.31}$$

Let $\Gamma_{r_1/T}(r_1 - r_2) = \mathbb{E}(V_{r_1}V_{r_2})$. We have

$$\begin{aligned} &\mathrm{Tr}\left(\Upsilon_T\widetilde{W}_b\right) \\ &= \sum_{1\leq r_1,\, r_2\leq T}\mathbb{E}\left(V_{r_1}V_{r_2}\right)\widetilde{K}_b\left(\frac{r_1}{T},\,\frac{r_2}{T}\right) \\ &= \sum_{r_2=1}^{T}\sum_{h=1-r_2}^{T-r_2}\Gamma_{r_2/T}(-h)\,\widetilde{K}_b\left(\frac{r_2+h}{T},\,\frac{r_2}{T}\right) \\ &= \left(\sum_{h=1}^{T-1}\sum_{r_2=1}^{T-h} + \sum_{h=1-T}^{0}\sum_{r_2=1-h}^{T}\right)\Gamma_{r_2/T}(-h)\,\widetilde{K}_b\left(\frac{r_2+h}{T},\,\frac{r_2}{T}\right). \end{aligned} \tag{S.32}$$

Using the Lipschitz property of $K(\cdot)$ and the fact that $\sup_{r_2,\,h}|\Gamma_{r_2/T}(-h)| < \infty$ which follows from

Assumption 4.1, some algebra shows that

$$\begin{aligned}
&\sum_{r_2=1}^{T-h} \Gamma_{r_2/T}(-h)\, \widetilde{K}_b\left(\frac{r_2+h}{T}, \frac{r_2}{T}\right) \qquad \text{(S.33)}\\
&= \sum_{r_2=1}^{T-h} \Gamma_{r_2/T}(-h)\, K_b\left(\frac{h}{T}\right) - \frac{1}{T}\sum_{r_2=1}^{T-h}\sum_{p=1}^{T} \Gamma_{r_2/T}(-h)\, K_b\left(\frac{r_2+h-p}{T}\right)\\
&\quad - \frac{1}{T}\sum_{r_2=1}^{T-h_1}\sum_{q=1}^{T} \Gamma_{r_2/T}(-h)\, K_b\left(\frac{q-r_2}{T}\right) + \sum_{r_2=1}^{T-h_1}\frac{1}{T^2}\sum_{p=1}^{T}\sum_{q=1}^{T} \Gamma_{r_2/T}(-h)\, K_b\left(\frac{p-q}{T}\right)\\
&= -\frac{1}{T}\sum_{r_2=1}^{T}\sum_{p=1}^{T} \Gamma_{r_2/T}(-h)\, K_b\left(\frac{r_2-p}{T}\right) - \frac{1}{T}\sum_{r_2=1}^{T}\sum_{q=1}^{T} \Gamma_{r_2/T}(-h)\, K_b\left(\frac{q-r_2}{T}\right)\\
&\quad + \sum_{r_2=1}^{T}\frac{1}{T^2}\sum_{p=1}^{T}\sum_{q=1}^{T} \Gamma_{r_2/T}(-h)\, K_b\left(\frac{p-q}{T}\right) + \sum_{r_2=1}^{T-h} \Gamma_{r_2/T}(-h)\, K_b\left(\frac{h}{T}\right) + O(|h|)\\
&= -\frac{1}{T}\sum_{r_2=1}^{T}\sum_{p=1}^{T} \Gamma_{r_2/T}(-h)\, K_b\left(\frac{r_2-p}{T}\right) + \sum_{r_2=1}^{T-h} \Gamma_{r_2/T}(-h)\, K_b\left(\frac{h}{T}\right) + O(|h|) + o(1)\\
&= \sum_{r_2=1}^{T} \Gamma_{r_2/T}(-h)\, K_b(0) - \frac{1}{T}\sum_{r_2=1}^{T}\sum_{p=1}^{T} \Gamma_{r_2/T}(-h)\, K_b\left(\frac{r_2-p}{T}\right)\\
&\quad + \sum_{r_2=1}^{T} \Gamma_{r_2/T}(-h)\left(K_b\left(\frac{h}{T}\right) - K_b(0)\right) + O(|h|) + o(1)\\
&= \sum_{r_2=1}^{T} \Gamma_{r_2/T}(-h)\, \widetilde{K}_b\left(\frac{r_2}{T}, \frac{r_2}{T}\right) + \sum_{r_2=1}^{T} \Gamma_{r_2/T}(-h)\left(K_b\left(\frac{h}{T}\right) - K_b(0)\right) + O(|h|) + o(1).
\end{aligned}$$

The same arguments yield

$$\begin{aligned}
&\sum_{r_2=1-h}^{T} \Gamma_{r_2/T}(-h)\, \widetilde{K}_b\left(\frac{r_2+h}{T}, \frac{r_2}{T}\right) \qquad \text{(S.34)}\\
&\quad = \sum_{r_2=1}^{T} \Gamma_{r_2/T}(-h)\, \widetilde{K}_b\left(\frac{r_2}{T}, \frac{r_2}{T}\right) + \sum_{r_2=1}^{T} \Gamma_{r_2/T}(-h)\left(K_b\left(\frac{h}{T}\right) - K_b(0)\right) + O(|h|) + o(1).
\end{aligned}$$

Using (S.33)-(S.34) into (S.32), we have

$$\begin{aligned}
&\mathrm{Tr}\left(\Upsilon_T \widetilde{W}_b\right) \qquad \text{(S.35)}\\
&= \sum_{h=-\infty}^{\infty}\sum_{r_2=1}^{T} \Gamma_{r_2/T}(-h)\, \widetilde{K}_b\left(\frac{r_2}{T}, \frac{r_2}{T}\right) + \sum_{h=-\infty}^{\infty}\sum_{r_2=1}^{T} \Gamma_{r_2/T}(-h)\left(K_b\left(\frac{h}{T}\right) - K_b(0)\right) + O(1)\\
&= \sum_{h=-\infty}^{\infty}\sum_{r_2=1}^{T} \Gamma_{r_2/T}(-h)\, \widetilde{K}_b\left(\frac{r_2}{T}, \frac{r_2}{T}\right) + (Tb)^{-q}\sum_{r_2=1}^{T}\sum_{h=-\infty}^{\infty} |h|^q\, \Gamma_{r_2/T}(-h)\left(\frac{K(h/Tb) - K(0)}{|h/(Tb)|^q}\right) + O(1)
\end{aligned}$$

$$= \sum_{h=-\infty}^{\infty} \sum_{r_2=1}^{T} \Gamma_{r_2/T}(-h) \widetilde{K}_b\left(\frac{r_2}{T}, \frac{r_2}{T}\right) + (Tb)^{-q} q_0 \sum_{r_2=1}^{T} \sum_{h=-\infty}^{\infty} |h|^q \Gamma_{r_2/T}(-h)(1+o(1)) + O(1).$$

By Theorem 2.1 in Casini (2023) and Lemma 4.1 in Casini, Deng and Perron (2023),

$$\sum_{h=-\infty}^{\infty} \Gamma_{s/T}(-h) = 2\pi f(s/T, 0)\left(1+O\left(\frac{1}{T}\right)\right), \tag{S.36}$$

and

$$\frac{1}{T} \sum_{h=-\infty}^{\infty} \sum_{s=1}^{T} \Gamma_{s/T}(-h) \widetilde{K}_b\left(\frac{s}{T}, \frac{s}{T}\right) = \int_0^1 \Omega(u) K_b^*(u, u)\, du + O\left(\frac{1}{T}\right), \tag{S.37}$$

where we have used $2\pi f(u, 0) = \Omega(u)$. Using Assumption 4.1 and (S.37), we yield

$$\begin{aligned} \mu_{b,T} &= \Omega^{-1} \int_0^1 \Omega(s) K_b^*(s, s)\, ds \\ &\quad + (Tb)^{-q} q_0 \left(\Omega_T^{-1} T^{-1} \sum_{r_2=1}^{T} \sum_{h=-\infty}^{\infty} |h|^q \Gamma_{r_2/T}(-h)\right)(1+o(1)) + O\left(\frac{1}{T}\right). \end{aligned} \tag{S.38}$$

Since $\mu_b = \Omega^{-1}\mathbb{E}(\mathscr{G}_b) = \Omega^{-1}\int_0^1 K_b^*(s, s)\,\Omega(s)\, ds$, b is fixed and $q \geq 1$, we have $\mu_{b,T} = \mu_b + O(T^{-1})$.

Next, we consider part (ii). For $m > 1$, let $r_{2m+1} = r_1$, $r_{2m+2} = r_2$ and $h_{m+1} = h_1$. Using the same argument used for the case $m = 1$ in (S.33) and using eq. (A.26) in Sun et al. (2008), we have

$$\begin{aligned} \operatorname{Tr}\left(\left(\Upsilon_T \widetilde{W}_b\right)^m\right) &= \sum_{r_1, r_2, \ldots, r_{2m+1}=1}^{T} \prod_{j=1}^{m} \mathbb{E}\left(V_{r_{2j-1}} V_{r_{2j}}\right) \widetilde{K}_b\left(\frac{r_{2j}}{T}, \frac{r_{2j+1}}{T}\right) \\ &= \sum_{r_1, r_2, \ldots, r_{2m+1}=1}^{T} \sum_{h_1=1-r_2}^{T-r_2} \sum_{h_2=1-r_4}^{T-r_4} \cdots \sum_{h_m=1-r_{2m}}^{T-r_{2m}} \prod_{j=1}^{m} \Gamma_{r_{2j}/T}(-h_j) \widetilde{K}_b\left(\frac{r_{2j}}{T}, \frac{r_{2j+1}+h_{j+1}}{T}\right) \\ &= \left(\sum_{h_1=1}^{T-1} \sum_{r_2=1}^{T-h_1} + \sum_{h_1=1-T}^{0} \sum_{r_2=1-h_1}^{T}\right) \cdots \left(\sum_{h_m=1}^{T-1} \sum_{r_{2m}=1}^{T-h_m} + \sum_{h_m=1-T}^{0} \sum_{r_{2m}=1-h_m}^{T}\right) \\ &\quad \times \prod_{j=1}^{m} \Gamma_{r_{2j}/T}(-h_j) \widetilde{K}_b\left(\frac{r_{2j}}{T}, \frac{r_{2j+1}+h_{j+1}}{T}\right) \\ &= L_1 + L_2, \end{aligned} \tag{S.39}$$

where

$$\begin{aligned} L_1 = &\left(\sum_{h_1=1}^{T-1} \sum_{r_2=1}^{T-h_1} + \sum_{h_1=1-T}^{0} \sum_{r_2=1-h_1}^{T}\right) \cdots \left(\sum_{h_m=1}^{T-1} \sum_{r_{2m}=1}^{T-h_m} + \sum_{h_m=1-T}^{0} \sum_{r_{2m}=1-h_m}^{T}\right) \\ &\prod_{j=1}^{m} \Gamma_{r_{2j}/T}(-h_j) \widetilde{K}_b\left(\frac{r_{2j}}{T}, \frac{r_{2j+1}+h_{j+1}}{T}\right), \end{aligned} \tag{S.40}$$

and

$$L_2 = O\left(\left(\sum_{h_1=1}^{T-1}\sum_{r_2=1}^{T-h_1} + \sum_{h_1=1-T}^{0}\sum_{r_2=1-h_1}^{T}\right)\cdots\left(\sum_{h_m=1}^{T-1}\sum_{r_{2m}=1}^{T-h_m} + \sum_{h_m=1-T}^{0}\sum_{r_{2m}=1-h_m}^{T}\right)\right. \\ \left.\prod_{j=1}^{m}\Gamma_{r_{2j}/T}(-h_j)\left(\frac{|h_{j+1}|}{Tb}\right)\right). \tag{S.41}$$

Using the same arguments as in eq. (A.30)-(A.31) in Sun et al. (2008), we can show that

$$L_1 = \sum_{h=-\infty}^{\infty}\sum_{r}\prod_{j=1}^{m}\Gamma_{r_{2j}/T}(-h_j)\left(\widetilde{K}_b\left(\frac{r_{2j}}{T},\frac{r_{2j+2}}{T}\right)\right) + O\left(2mT^{m-2}\left(\overline{C}_1 b\right)^{m-2}\right),$$

where $O(2mT^{m-2}((\overline{C}_1 b))^{m-2})$ follows from

$$\begin{aligned}
&\sum_{h_1=-\infty}^{\infty}\sum_{r_2=1}^{T}\cdots\sum_{h_a=-\infty}^{\infty}\sum_{r_{2a}=1}^{-h_a}\cdots\sum_{h_m=-\infty}^{\infty}\sum_{r_{2m}=1}^{T}\prod_{j=1}^{m}\left(\sup_{r_{2j}}\left|\Gamma_{r_{2j}/T}(-h_j)\right|\right)|h_a|\prod_{j\neq a}\left|\widetilde{K}_b\left(\frac{r_{2j}}{T},\frac{r_{2j+2}}{T}\right)\right| \\
&\le\left[\sup_t\sum_{s=1}^{T}\widetilde{K}_b\left(\frac{s}{T},\frac{t}{T}\right)\right]^{m-2}\left(\sum_{h=-\infty}^{\infty}\sup_s\left|\Gamma_{s/T}(-h)\right|\right)^{m-1}\left(\sum_{h_a=-\infty}^{\infty}\sup_s\left|\Gamma_{s/T}(-h_a)\right||h_a|\right) \\
&\le O\left(2mT^{m-2}\left(\overline{C}_1 b\right)^{m-2}\right),
\end{aligned}$$

uniformly over m for some integer a such that $1 \le a \le m$. A similar argument yields that $L_2 = o(2mT^{m-2}(\overline{C}_1 b)^{m-2})$ uniformly over m. Thus,

$$\mathrm{Tr}\left(\left(\Upsilon_T\widetilde{W}_b\right)^m\right) = \sum_{h=-\infty}^{\infty}\sum_{r}\prod_{j=1}^{m}\Gamma_{r_{2j}/T}(-h_j)\left(\widetilde{K}_b\left(\frac{r_{2j}}{T},\frac{r_{2j+2}}{T}\right)\right) + O\left(2mT^{m-2}\left(\overline{C}_1 b\right)^{m-2}\right),$$

and using $\tau_1 = \tau_{m+1}$ we yield

$$\begin{aligned}
\kappa_{m,T} &= 2^{m-1}(m-1)!T^{-m}\Omega_T^{-m}\mathrm{Tr}\left((\Upsilon_T A_T W_b A_T)^m\right) \\
&= 2^{m-1}(m-1)!\Omega_T^{-m}\left(T^{-m}\sum_r\Omega(r_{2j}/T)\widetilde{K}_b\left(\frac{r_{2j}}{T},\frac{r_{2j+2}}{T}\right) + O\left(2mT^{-2}\left(\overline{C}_1 b\right)^{m-2}\right)\right) \\
&= 2^{m-1}(m-1)!\Omega^{-m}\left(\int_0^1\cdots\int_0^1\left(\prod_{j=1}^{m}\Omega(\tau_j)K_b^*(\tau_j,\tau_{j+1})\right)d\tau_1\cdots d\tau_m + O\left(2mT^{-2}\left(\overline{C}_1 b\right)^{m-2}\right)\right) \\
&= \kappa_m + O\left(2mT^{-2}\left(\overline{C}_1 b\right)^{m-2}\right),
\end{aligned} \tag{S.42}$$

uniformly over m.

Next, we consider part (iii). From (S.17) and part (ii), we have uniformly over m,

$$\Xi_{m,T} = \Xi_m + O\left(\frac{m!2^m}{T^2}\left(\overline{C}_1 b\right)^{m-2}\sum_\pi\frac{m!}{m_1!m_2!\cdots m_k!}\right) \tag{S.43}$$

$$= \Xi_m + O\left(\frac{m!2^{2m}}{T^2}\left(\overline{C}_1 b\right)^{m-2}\right),$$

where we have used $\sum_\pi \frac{m!}{m_1!m_2!\cdots m_k!} < 2^m$. $\square$

Proof of Theorem 4.1. Let $F_\chi^{(m)}(\cdot)$ denote the mth derivative of $F_\chi(\cdot)$. Since $F_\chi(\cdot)$ is a bounded function, we can write

$$\begin{aligned}
\mathbb{P}\left(\left|\frac{\int_0^1 \Sigma(u)\, dW_1(u)}{\sqrt{\mathscr{G}_b}}\right| \le z\right) &= \lim_{C\to\infty} \mathbb{E}\left(F_\chi\left(z^2\mathscr{G}_b/\Omega\right)\mathbf{1}\left(|\mathscr{G}_b - \mu_b| \le \Omega C\right)\right) \qquad \text{(S.44)}\\
&= \lim_{C\to\infty} \mathbb{E}\sum_{m=1}^{\infty}\frac{1}{m!}F_\chi^{(m)}\left(\mu_b z^2/\Omega\right)\Omega^{-m}\left(\mathscr{G}_b - \mu_b\right)^m z^{2m}\mathbf{1}\left\{|\mathscr{G}_b - \mu_b| \le \Omega C\right\}\\
&= \lim_{C\to\infty}\sum_{m=1}^{\infty}\frac{1}{m!}F_\chi^{(m)}\left(\mu_b z^2/\Omega\right)\Xi_m z^{2m}\mathbf{1}\left\{|\mathscr{G}_b - \mu_b| \le \Omega C\right\},
\end{aligned}$$

where $\Xi_m = \Omega^{-m}\mathbb{E}((\mathscr{G}_b - \mu_b)^m)$. Since $F_\chi(z^2)$ decays exponentially as $z \to \infty$, there exists a constant $C_2 > 0$ such that $|F_\chi^{(m)}(\mu_b z^2/\Omega)z^{2m}| < C_2$ for all m. Using Lemma S.A.2, we yield

$$\begin{aligned}
\left|\sum_{m=1}^{\infty}\frac{1}{m!}F_\chi^{(m)}\left(\mu_b z^2/\Omega\right)\Xi_m z^{2m}\right| &\le C_2\sum_{m=1}^{\infty}\frac{1}{m!}\Xi_m \le C_2 D\sum_{m=1}^{\infty}\frac{1}{m!}2^{2m}m!C_\Omega^m\left(\overline{C}_1 b\right)^{m-1} \qquad \text{(S.45)}\\
&= C_2 D\left(\overline{C}_1 b\right)^{-1}\sum_{m=1}^{\infty}\left(4C_\Omega\overline{C}_1 b\right)^m,
\end{aligned}$$

where $D_m \le D$ for some $D < \infty$. The right-hand side of (S.45) is bounded in view of $b < 1/(4C_\Omega\overline{C}_1)$. This implies that

$$\mathbb{P}\left(\left|\frac{\int_0^1 \Sigma(u)\, dW_1(u)}{\sqrt{\mathscr{G}_b}}\right| \le z\right) = \sum_{m=1}^{\infty}\frac{1}{m!}F_\chi^{(m)}\left(\mu_b z^2/\Omega\right)\Xi_m z^{2m}, \qquad \text{(S.46)}$$

provided that $b < 1/(4C_\Omega\overline{C}_1)$.

From (S.25) we have

$$Z_{T,0}(z) = \mathbb{P}\left(\left|\frac{\sqrt{T}\left(\widehat{\beta} - \beta_0\right)}{\sqrt{\widehat{\Omega}_b}}\right| \le z\right) = \mathbb{E}\left(F_\chi\left(z^2\zeta_{b,T}\right)\right) + O\left(T^{-1}\right). \qquad \text{(S.47)}$$

Using a similar argument as in (S.44),

$$\mathbb{E}\left(F_\chi\left(z^2\zeta_{b,T}\right)\right) - \sum_{m=1}^{\infty}\frac{1}{m!}F_\chi^{(m)}\left(\mu_{b,T}z^2\right)\Xi_{m,T}z^{2m} \to 0, \qquad \text{(S.48)}$$

uniformly over T since by Lemma S.A.3-(iii) we have

$$\Xi_{m,T} = \Xi_m + O\left(\frac{2^{2m} m!}{T^2}\left(\overline{C}_1 b\right)^{m-2}\right),$$

uniformly in m and $|F_\chi^{(m)}(\mu_{b,T} z^2) z^{2m}| < C_2$ for some constant $C_2 > 0$ for all m so that

$$\left|\sum_{m=1}^{\infty} \frac{1}{m!} F_\chi^{(m)}\left(\mu_{b,T} z^2\right) \Xi_{m,T} z^{2m}\right| \leq C_2 \sum_{m=1}^{\infty} \frac{1}{m!} |\Xi_m| + \frac{C_2}{T^2} \sum_{m=1}^{\infty} 2^{2m}\left(\overline{C}_1 b\right)^{m-2} < \infty,$$

provided that $b < 1/(4\overline{C}_1)$. Note that $b < 1/(16 C_{2,\Omega} \int_{-\infty}^{\infty} |k(x)| dx) < 1/(4\overline{C}_1)$ by assumption. It follows that

$$Z_{T,0}(z) = \sum_{m=1}^{\infty} \frac{1}{m!} F_\chi^{(m)}\left(\mu_{b,T} z^2\right) \Xi_{m,T} z^{2m} + O\left(T^{-1}\right), \tag{S.49}$$

uniformly over $z \in \mathbb{R}_+$.

By Lemma S.A.3-(i), we have

$$F_\chi^{(m)}\left(\mu_{b,T} z^2\right) = F_\chi^{(m)}\left(\mu_b z^2/\Omega\right) + O\left(F_\chi^{(m+1)}\left(\mu_b z^2/\Omega\right) z^2 T^{-1}\right). \tag{S.50}$$

Combining (S.46) and (S.49)-(S.50) leads to

$$\begin{aligned}
&|Z_{T,0}(z) - Z_0(z)| \qquad\qquad (\text{S.51})\\
&= \left|\sum_{m=1}^{\infty} \frac{1}{m!} F_\chi^{(m)}\left(\mu_{b,T} z^2\right) \Xi_{m,T} z^{2m} - \sum_{m=1}^{\infty} \frac{1}{m!} F_\chi^{(m)}\left(\mu_b z^2\right) \Xi_m z^{2m}\right| + O\left(\frac{1}{T}\right)\\
&= \left|\sum_{m=1}^{\infty} \frac{1}{m!} F_\chi^{(m)}\left(\mu_b z^2\right) z^{2m}\left(\Xi_{m,T} - \Xi_m\right)\right| + O\left(\frac{1}{T}\right)\\
&= \left|\sum_{m=1}^{\infty} \frac{1}{m!} F_\chi^{(m)}\left(\mu_b z^2\right) z^{2m} O\left(\frac{m! 2^{2m} (\overline{C}_1 b)^{m-2}}{T^2}\right)\right| + O\left(\frac{1}{T}\right)\\
&= O\left(\frac{1}{T^2} \sum_{m=1}^{\infty} 2^{2m} (\overline{C}_1 b)^{m-2}\right) + O\left(\frac{1}{T}\right)\\
&= O\left(\frac{1}{T}\right).
\end{aligned}$$

uniformly over $z \in \mathbb{R}$ where we have used Lemma S.A.3-(iii). Hence, $D_1 = O\left(T^{-1}\right)$.

Let $\widehat{\Omega} = \int_0^1 \widehat{\Omega}(u) du$ where $\widehat{\Omega}(u)$ was defined in 4.3. Note that $\widehat{\mathscr{G}}_b = \mathscr{G}_b + O((T h_1 h_2)^{-1/2})$ by definition of $\widehat{\Omega}(u)$. Using this and proceeding as in (S.44), we yield

$$\begin{aligned}
\widehat{Z}_0(z) &= \mathbb{P}\left(\left|\frac{\int_0^1 \widehat{\Sigma}(u) dW_p(u)}{\sqrt{\widehat{\mathscr{G}}_b}}\right| \leq z\right) \qquad (\text{S.52})\\
&= \lim_{C \to \infty} \mathbb{E}\left(F_\chi\left(z^2 \widehat{\mathscr{G}}_b / \widehat{\Omega}\right) \mathbf{1}\left(\left|\widehat{\mathscr{G}}_b - \widehat{\mu}_b\right| \leq \widehat{\Omega} C\right)\right)
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{C\to\infty} \mathbb{E} \sum_{m=1}^{\infty} \frac{1}{m!} F_{\chi}^{(m)} \left(\widehat{\mu}_b z^2 / \widehat{\Omega} \right) \widehat{\Omega}^{-m} \left(\widehat{\mathscr{G}}_b - \widehat{\mu}_b \right)^m z^{2m} \mathbf{1} \left\{ \left| \widehat{\mathscr{G}}_b - \widehat{\mu}_b \right| \leq \Omega C \right\} \\
&= \lim_{C\to\infty} \mathbb{E} \sum_{m=1}^{\infty} \frac{1}{m!} F_{\chi}^{(m)} \left(\mu_b z^2 / \Omega \right) \Omega^{-m} \left(\mathscr{G}_b - \mu_b \right)^m z^{2m} \mathbf{1} \left\{ \left| \mathscr{G}_b - \mu_b \right| \leq \Omega C \right\} + O((Th_1h_2)^{-1/2}) \\
&= \lim_{C\to\infty} \sum_{m=1}^{\infty} \frac{1}{m!} F_{\chi}^{(m)} \left(\mu_b z^2 / \Omega \right) \Xi_m z^{2m} \mathbf{1} \left\{ \left| \mathscr{G}_b - \mu_b \right| \leq \Omega C \right\} + O\left((Th_1h_2)^{-1/2} \right),
\end{aligned}
$$

uniformly in $z \in \mathbb{R}_+$. This implies $D_2 = O((Th_1h_2)^{-1/2})$. $\square$

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