Supplement for Online Publication to "Theory of Evolutionary Spectra for Heteroskedasticity and Autocorrelation Robust Inference in Possibly Misspecified and Nonstationary Models"

Alessandro Casini*

University of Rome Tor Vergata

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Abstract

This supplemental material is for online publication only. It contains the proofs of the results of Section 3 in the paper.

S.A Appendix: Proofs of the Results of Section 3

In the proofs below, we discard the degrees of freedom adjustment $T/(T-p)$ from the derivations since asymptotically it does not play any role. Similarly, we use T/n_T in place of $(T-n_T)/n_T$ in the expression for $\widehat{\Gamma}(k)$. In some of the proofs below we first consider the locally stationary case under Assumption S.A.1-S.A.2 and then extend the results to the SLS case. Note that Assumption S.A.1-S.A.2 are implied by Assumption 3.1-3.2 since the former are weaker because local stationarity does not allow for break points in the spectrum. A function $G(\cdot,\cdot):[0,1]\times\mathbb{R}\rightarrow\mathbb{C}$ is said to be right-differentiable at u_0 if $\partial G(u_0,\omega)/\partial_+u\triangleq\lim_{u\rightarrow u_0^+}(G(u_0,\omega)-G(u,\omega))/(u-u_0)$ exists for any $\omega\in\mathbb{R}$. We sometimes use $\sum_t$ omitting the limits of the summation for the sum in $\widetilde{c}_T(u,k)$.

Assumption S.A.1. *$\{V_{t,T}\}$ is a mean-zero locally stationary process, $A(u,\omega)$ is twice differentiable in u with uniformly bounded and Lipschitz continuous derivatives $(\partial/\partial u)A(u,\cdot)$ and $(\partial^2/\partial u^2)A(u,\cdot)$, and Lipschitz continuous in the second component with index $\vartheta=1$.*

Assumption S.A.2. *(i) $\sum_{k=-\infty}^{\infty}\sup_{u\in[0,1]}\|c(u,k)\|<\infty$, $\sum_{k=-\infty}^{\infty}\sup_{u\in[0,1]}\left\|(\partial^2/\partial u^2)c(u,k)\right\|<\infty$ and $\sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\sup_{u\in[0,1]}\kappa_{V,\lfloor Tu\rfloor}^{(a,b,c,d)}(k,j,l)<\infty$ for all $a,b,c,d\leq p$; (ii) For all $a,b,c,d\leq p$ there exists a function $\widetilde{\kappa}_{a,b,c,d}:[0,1]\times\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}\rightarrow\mathbb{R}$ such that $\sup_{u\in[0,1]}|\kappa_{V,\lfloor Tu\rfloor}^{(a,b,c,d)}(k,s,l)-\widetilde{\kappa}_{a,b,c,d}(u,k,s,l)|\leq KT^{-1}$ for some constant K; the function $\widetilde{\kappa}_{a,b,c,d}(u,k,s,l)$ is twice differentiable in u with uniformly bounded and Lipschitz continuous derivative $(\partial^2/\partial u^2)\widetilde{\kappa}_{a,b,c,d}(u,\cdot,\cdot,\cdot)$.*

*Department of Economics and Finance, University of Rome Tor Vergata, Via Columbia 2, Rome 00133, IT. Email: alessandro.casini@uniroma2.it.

S.A.1 Preliminary Lemmas

Lemma S.A.1. *Under Assumption 3.1-3.2,*

$$\sup_{u\in\{\{(0,1)\}/\{\lambda_j^0,\, j=1,\ldots,\, m_0\}\}} \sup_{k\in\mathbb{Z}} \left\| \mathrm{Cov}\left(V_{\lfloor Tu\rfloor,T},\, V_{\lfloor Tu\rfloor-k,T}\right) - c\,(u,\, k)\right\| = O(T^{-1}),$$

$$\sup_{u\in(0,\,1)} \sup_{k\geq 0} \left\| \mathrm{Cov}\left(V_{\lfloor Tu\rfloor,T},\, V_{\lfloor Tu\rfloor-k,T}\right) - c\,(u,\, k)\right\| = O(T^{-1}),$$

$$\max_{u=\lambda_j^0,\, j=1,\ldots,\, m_0} \sup_{k<0} \left\| \mathrm{Cov}\left(V_{\lfloor Tu\rfloor,T},\, V_{\lfloor Tu\rfloor-k,T}\right) - c\,(u,\, -k)\right\| = O(T^{-1}).$$

Proof of Lemma S.A.1. It is sufficient to consider the scalar case $p=1$. Consider first $Tu \notin \mathcal{T}$, $\lambda_{j-1}^0 < u < \lambda_j^0$. Using the spectral representation (2.2), (2.4) and Assumption 2.1 leads to

$$\begin{aligned}\mathrm{Cov}\left(V_{\lfloor Tu\rfloor,T},\, V_{\lfloor Tu\rfloor-k,T}\right) &= \int_{-\pi}^{\pi} \exp\left(i\omega k\right) A^0_{j,\lfloor Tu\rfloor,T}\left(\omega\right) A^0_{j,\lfloor Tu\rfloor-k,T}\left(-\omega\right) d\omega \\ &= \int_{-\pi}^{\pi} \exp\left(i\omega k\right) A_j\left(u,\,\omega\right) A_j\left(u-k/T,\,-\omega\right) d\omega + O(T^{-1}) \\ &= c\,(u,\,k) + O(T^{-1}), \end{aligned} \tag{S.1}$$

where the $O(T^{-1})$ term is uniform in $u \in \{\{(0,\,1)\}/\{\lambda_j^0,\, j=1,\ldots,\, m_0\}\}$ and k. Now consider the case $Tu \in \mathcal{T}$, $u = T_j^0/T$ and $k \geq 0$. Using (2.2) and (2.4) yields

$$\begin{aligned}\mathrm{Cov}\left(V_{\lfloor Tu\rfloor,T},\, V_{\lfloor Tu\rfloor-k,T}\right) &= \int_{-\pi}^{\pi} \exp\left(i\omega k\right) A^0_{j,\lfloor Tu\rfloor,T}\left(\omega\right) A^0_{j+1,\lfloor Tu\rfloor-k,T}\left(-\omega\right) d\omega \\ &= \int_{-\pi}^{\pi} \exp\left(i\omega k\right) A_j\left(u,\,\omega\right) A_{j+1}\left(u-k/T,\,-\omega\right) d\omega + O(T^{-1}) \\ &= c\,(u,\,-k) + O(T^{-1}), \end{aligned} \tag{S.2}$$

where the $O\left(T^{-1}\right)$ term is uniform in u and $k \geq 0$. The argument for the case $Tu \in \mathcal{T}$ and $k<0$ is the same as for the case $Tu \notin \mathcal{T}$. $\square$

Lemma S.A.2. *Under Assumption* S.A.1-S.A.2, $\sup_{u\in(0,1)} \sup_{v,\,k\in\mathbb{Z}} ||\Gamma_u\left(v\right) - \Gamma_{u+k/T}\left(v\right)|| = O(T^{-1})$.

Proof of Lemma S.A.2. We know that $\Gamma_u\left(v\right) = c\left(u,\, v\right) + O(T^{-1})$ uniformly in u and v by Lemma S.A.1 where $c\left(u,\, v\right) = \int_{-\pi}^{\pi} e^{i\omega v} f\left(u,\,\omega\right) d\omega$. Using Assumption S.A.1,

$$\begin{aligned} c\left(u,\,v\right) &= \int_{-\pi}^{\pi} e^{i\omega v} f\left(u+k/T,\,\omega\right) d\omega + O\left(k/T\right) \\ &= c\left(u+k/T,\,v\right) + O\left(k/T\right) \\ &= \Gamma_{u+k/T}\left(v\right) + O\left(k/T\right) + O(T^{-1}), \end{aligned}$$

uniformly in $u \in (0,\,1)$ and $v,\, k \in \mathbb{Z}$. $\square$

Let

$$\mathrm{MSE}\left(\widetilde{c}_T\left(u_0,\,k\right)\right) = Tb_{2,T}\mathbb{E}\left[\mathrm{vec}\left(\widetilde{c}_T\left(u_0,\,k\right) - c\left(u_0,\,k\right)\right)' W \mathrm{vec}\left(\widetilde{c}_T\left(u_0,\,k\right) - c\left(u_0,\,k\right)\right)\right],$$

where W is some $p^2 \times p^2$ weight matrix.

Lemma S.A.3. *Suppose Assumption S.A.1-S.A.2 hold and $b_{2,T} \to 0$ as $T \to \infty$. Then, for all $u_0 \in (0,\, 1)$,*

$$\mathbb{E}\left[\widetilde{c}_T\left(u_0,\, k\right)\right] = c\left(u_0,\, k\right) + \frac{1}{2} b_{2,T}^2 \int_0^1 x^2 K_2\left(x\right) dx \left[\frac{\partial^2}{\partial^2 u} c\left(u_0,\, k\right)\right] + o\left(b_{2,T}^2\right) + O\left(1/\left(T b_{2,T}\right)\right), \tag{S.3}$$

and for all $j,\, l,\, r,\, w \leq p$,

$$\begin{aligned} &\mathrm{Cov}\left[\widetilde{c}_T^{(j,l)}\left(u_0,\, k\right),\, \widetilde{c}_T^{(r,w)}\left(u_0,\, k\right)\right] \\ &= \frac{1}{T b_{2,T}} \int_0^1 K_2^2\left(x\right) dx \sum_{l=-\infty}^{\infty} \left[c^{(j,r)}\left(u_0,\, l\right) c^{(l,w)}\left(u_0,\, l\right) + c^{(j,w)}\left(u_0,\, l\right) c^{(l,r)}\left(u_0,\, l+2k\right)\right] \\ &\quad + \frac{1}{T b_{2,T}} \int_0^1 K_2^2\left(x\right) dx \sum_{h_1=-\infty}^{\infty} \widetilde{\kappa}_{j,l,r,w}\left(u_0,\, -k,\, h_1,\, h_1 - k\right) + o\left(\frac{1}{T b_{2,T}}\right). \end{aligned} \tag{S.4}$$

If $T b_{2,T}^5 \to \eta \in (0,\, \infty)$, then, for all $u_0 \in (0,\, 1)$, $\widetilde{c}_T\left(u_0,\, k\right) - c\left(u_0,\, k\right) = O_{\mathbb{P}}\left(\sqrt{T b_{2,T}}\right)$.

If in addition $V_{t,T}$ is Gaussian, then for all $u_0 \in (0,\, 1)$,

$$\begin{aligned} &\mathrm{Cov}\left[\widetilde{c}_T^{(j,l)}\left(u_0,\, k\right),\, \widetilde{c}_T^{(r,w)}\left(u_0,\, k\right)\right] \\ &= \frac{1}{T b_{2,T}} \int_0^1 K_2^2\left(x\right) dx \sum_{l=-\infty}^{\infty} \left[c^{(j,r)}\left(u_0,\, l\right) c^{(l,w)}\left(u_0,\, l\right) + c^{(j,w)}\left(u_0,\, l\right) c^{(l,r)}\left(u_0,\, l+2k\right)\right] \\ &\quad + o\left(1/\left(T b_{2,T}\right)\right), \end{aligned} \tag{S.5}$$

and if $T b_{2,T}^5 \to \eta \in (0,\, \infty)$, then

$$\begin{aligned} \lim_{T\to\infty} \mathrm{MSE}\left(\widetilde{c}_T\left(u_0,\, k\right)\right) &= \frac{\eta}{4} \left(\int_0^1 x^2 K_2\left(x\right) dx\right)^2 \left[\frac{\partial^2}{\partial^2 u} \mathrm{vec}\left(c\left(u_0,\, k\right)\right)\right]' W \left[\frac{\partial^2}{\partial^2 u} \mathrm{vec}\left(c\left(u_0,\, k\right)\right)\right] \\ &\quad + \int_0^1 K_2^2\left(x\right) dx\, \mathrm{tr} W \sum_{l=-\infty}^{\infty} \mathrm{vec}\left(c\left(u_0,\, l\right)\right) \left[\mathrm{vec}\left(c\left(u_0,\, l\right)\right)' + \mathrm{vec}\left(c\left(u_0,\, l+2k\right)\right)'\right]. \end{aligned}$$

Proof of Lemma S.A.3. The bias expression follows from Dahlhaus (1997). For the second moment and MSE of $\widetilde{c}_T\left(u_0,\, k\right)$, we first present the proof for the case where $V_{t,T}$ is Gaussian and $p = 1$. Evaluating the expectation, we have for $k < 0$,

$$\begin{aligned} &\mathrm{Var}\left[\widetilde{c}_T\left(u_0,\, k\right)\right] \\ &= \frac{1}{\left(T b_{2,T}\right)^2} \sum_t \sum_s K_2^*\left(\frac{u_0 - \left(t + k/2\right)/T}{b_T}\right) K_2^*\left(\frac{u_0 - \left(s + k/2\right)/T}{b_T}\right) \mathbb{E}\left(V_{t,T} V_{s,T}\right) \mathbb{E}\left(V_{t+k,T} V_{s+k,T}\right) \\ &\quad + \frac{1}{\left(T b_{2,T}\right)^2} \sum_t \sum_s K_2^*\left(\frac{u_0 - \left(t + k/2\right)/T}{b_T}\right) K_2^*\left(\frac{u_0 - \left(s + k/2\right)/T}{b_T}\right) \mathbb{E}\left(V_{t,T} V_{t+k,T}\right) \mathbb{E}\left(V_{s,T} V_{s+k,T}\right) \\ &\quad + \frac{1}{\left(T b_{2,T}\right)^2} \sum_t \sum_s K_2^*\left(\frac{u_0 - \left(t + k/2\right)/T}{b_T}\right) K_2^*\left(\frac{u_0 - \left(s + k/2\right)/T}{b_T}\right) \mathbb{E}\left(V_{t,T} V_{s+k,T}\right) \mathbb{E}\left(V_{s,T} V_{t+k,T}\right) \\ &\quad - \left[\mathbb{E}\left(\widetilde{c}_T\left(u_0,\, k\right)\right)\right]^2. \end{aligned}$$

Using the continuity of K_2, $\left(s - t\right)/T \to 0$ for fixed s and t, the smoothness of $\Gamma_u\left(\cdot\right)$ and Lemma S.A.1, implies that the first term on the right-hand side is equal to

$$\frac{1}{Tb_{2,T}}\int_0^1 x^2K_2(x)\,dx\sum_{l=-\infty}^{\infty}c(u_0,\,l)^2.$$

For the second and third terms we use a similar argument with in addition Lemma 6.2.1 in Fuller (1995) so that

$$\begin{aligned}&\mathrm{Var}\left[\widetilde{c}_T(u_0,\,k)\right]\\&\quad=\frac{1}{Tb_{2,T}}\int_0^1 x^2K_2(x)\,dx\sum_{l=-\infty}^{\infty}c(u_0,\,l)^2+\frac{1}{Tb_{2,T}}\int_0^1 x^2K_2(x)\,dx\sum_{l=-\infty}^{\infty}c(u_0,\,l)\,c(u_0,\,l+2k)+o\left(\frac{1}{Tb_{2,T}}\right).\end{aligned}\tag{S.6}$$

Next, (S.5) follows similarly. We have

$$\begin{aligned}&\mathrm{Cov}\left[\widetilde{c}_T^{(j,l)}(u_0,\,k),\,\widetilde{c}_T^{(r,w)}(u_0,\,k)\right]\\&\quad=\frac{1}{(Tb_{2,T})^2}\sum_t\sum_s K_2^*\left(\frac{u_0-(t+k/2)/T}{b_T}\right)K_2^*\left(\frac{u_0-(s+k/2)/T}{b_T}\right)\mathbb{E}\left(V_{t,T}^{(j)}V_{s,T}^{(r)}\right)\mathbb{E}\left(V_{t+k,T}^{(l)}V_{s+k,T}^{(w)}\right)\\&\qquad+\frac{1}{(Tb_{2,T})^2}\sum_t\sum_s K_2^*\left(\frac{u_0-(t+k/2)/T}{b_T}\right)K_2^*\left(\frac{u_0-(s+k/2)/T}{b_T}\right)\mathbb{E}\left(V_{t,T}^{(j)}V_{s+k,T}^{(w)}\right)\mathbb{E}\left(V_{t+k,T}^{(l)}V_{s,T}^{(r)}\right)\\&\quad=\frac{1}{Tb_{2,T}}\int_0^1 K_2^2(x)\,dx\sum_{l=-\infty}^{\infty}\left[c^{(j,r)}(u_0,\,l)\,c^{(l,w)}(u_0,\,l)+c^{(j,w)}(u_0,\,l)\,c^{(l,r)}(u_0,\,l+2k)\right]+o\left(1/(Tb_{2,T})\right).\end{aligned}$$

Using a standard bias-variance argument, we have $\widetilde{c}_T(u_0,\,k)-c(u_0,\,k)=o_{\mathbb{P}}(1)$. If $Tb_{2,T}^5\to\eta\in(0,\,\infty)$, the asymptotic MSE of $\widetilde{c}_T(u_0,\,k)$ is given by

$$\begin{aligned}\lim_{T\to\infty}\mathrm{MSE}\left(\widetilde{c}_T(u_0,\,k)\right)=&\ \frac{\eta}{4}\left(\int_0^1 x^2K_2(x)\,dx\right)^2\left[\frac{\partial^2}{\partial^2 u}c(u_0,\,k)\right]^2\\&+\int_0^1 K_2^2(x)\,dx\sum_{l=-\infty}^{\infty}c(u_0,\,l)\left[c(u_0,\,l)+c(u_0,\,l+2k)\right].\end{aligned}\tag{S.7}$$

The latter suggests that if $Tb_{2,T}^5\to\eta\in(0,\,\infty)$, then $\widetilde{c}_T(u_0,\,k)-c(u_0,\,k)=O_{\mathbb{P}}\left(\sqrt{Tb_{2,T}}\right)$ for all $u_0\in(0,\,1)$. The MSE expression for the multivariate case follows from (S.7).

Consider now the second moment of $\widetilde{c}_T(u_0,\,k)$ for the general case. When $V_{t,T}$ is non-Gaussian, there is an extra term in $\mathrm{Cov}[\widetilde{c}_T^{(j,l)}(u_0,\,k),\,\widetilde{c}_T^{(r,w)}(u_0,\,k)]$, namely

$$\frac{1}{(Tb_{2,T})^2}\sum_t\sum_s K_2^*\left(\frac{u_0-(t+k/2)/T}{b_T}\right)K_2^*\left(\frac{u_0-(s+k/2)/T}{b_T}\right)\kappa_{V,t}^{(j,l,r,w)}(-k,\,s-t,\,s-t-k).$$

By Assumption S.A.2 with $u=t/T$,

$$\sup_{u\in(0,\,1)}\left|\kappa_{V,Tu}^{(j,l,r,w)}(-k,\,s-Tu,\,s-Tu-k)-\widetilde{\kappa}_{j,l,r,w}(u,\,-k,\,s-Tu,\,s-Tu-k)\right|=O(T^{-1}).$$

Taking a second-order Taylor's expansion of $\kappa_{V,Tu}^{(j,l,r,w)}$ around u_0 we have

$$\begin{aligned}
&\frac{1}{(Tb_{2,T})^2}\sum_t\sum_s K_2^*\left(\frac{u_0-(t+k/2)/T}{b_T}\right)K_2^*\left(\frac{u_0-(s+k/2)/T}{b_T}\right)\kappa_{V,Tu}^{(j,l,r,w)}(-k,\,s-Tu,\,s-Tu-k)\\
&\quad=\frac{1}{(Tb_{2,T})^2}\sum_t\sum_s K_2^*\left(\frac{u_0-(t+k/2)/T}{b_T}\right)K_2^*\left(\frac{u_0-(s+k/2)/T}{b_T}\right)\\
&\quad\times\widetilde{\kappa}_{j,l,r,w}(u_0,\,-k,\,s-Tu_0,\,s-Tu_0-k)\\
&\quad+\frac{1}{(Tb_{2,T})^2}\sum_t\sum_s K_2^*\left(\frac{u_0-(t+k/2)/T}{b_T}\right)K_2^*\left(\frac{u_0-(s+k/2)/T}{b_T}\right)\\
&\quad\times\frac{\partial\widetilde{\kappa}_{j,l,r,w}}{\partial u}(u_0,\,-k,\,s-Tu_0,\,s-Tu_0-k)(u_0-u)\\
&\quad+\frac{1}{(Tb_{2,T})^2}\sum_t\sum_s K_2^*\left(\frac{u_0-(t+k/2)/T}{b_T}\right)K_2^*\left(\frac{u_0-(s+k/2)/T}{b_T}\right)\\
&\quad\times\frac{\partial^2\widetilde{\kappa}_{j,l,r,w}}{\partial u^2}(u_0,\,-k,\,s-Tu_0,\,s-Tu_0-k)(u_0-u)^2\\
&\quad=\frac{1}{Tb_{2,T}}\int_0^1 K_2^2(x)\,dx\sum_{h_1=-\infty}^{\infty}\widetilde{\kappa}_{j,l,r,w}(u_0,\,-k,\,h_1,\,h_1-k)+o\left(\frac{1}{Tb_{2,T}}\right).\square
\end{aligned}$$

Lemma S.A.4. *Suppose Assumption 3.1-3.2 hold and* $b_{2,T}\to 0$ *as* $T\to\infty$. *For each* $T\lambda_j^0=Tu_0\in\mathcal{T}$ $(j=1,\ldots,m_0)$ *and* $|k|/Tb_{2,T}\to\eta_2\in(0,\,\lambda_{j+1}^0-\lambda_j^0)$,

$$\begin{aligned}
\mathbb{E}[\widetilde{c}_T(u_0,\,k)]&=c(u_0,\,k)+\frac{1}{2}b_{2,T}^2\int_0^1 x^2K_2(x)\,dx\\
&\quad\times\int_{-\pi}^{\pi}\exp(i\omega k)(C_1(u_0,\,\omega)+C_2(u_0,\,\omega)+C_3(u_0,\,\omega))\,d\omega+O\left(\frac{1}{Tb_{2,T}}\right)+o\left(b_{2,T}^2\right),
\end{aligned}$$

where

$$C_1(u_0,\,\omega)=2\frac{\partial A_j(u_0,\,-\omega)}{\partial_-u}\frac{\partial A_{j+1}(v_0,\,\omega)}{\partial_+v},\qquad C_2(u_0,\,\omega)=\frac{\partial^2A_{j+1}(v_0,\,\omega)}{\partial_+v^2}A_j(u_0,\,-\omega)$$

$$C_3(u_0,\,\omega)=\frac{\partial^2A_j(u_0,\,\omega)}{\partial_-u^2}A_{j+1}(v_0,\,\omega),$$

and $v_0=u_0-k/2T$. *For* $Tu_0\notin\mathcal{T}$ *or for* $Tu_0\in\mathcal{T}$ *and* $|k|/Tb_{2,T}\to 0$, *(S.3) and (S.4) hold. For all* $u_0\in(0,\,1)$, $\lim_{T\to\infty}b_{2,T}^{-2}\,\mathbb{E}[\widetilde{c}_T(u_0,\,k)-c(u_0,\,k)]<\infty$, *and if further it holds that* $Tb_{2,T}^5\to\eta\in(0,\,\infty)$, *then* $\lim_{T\to\infty}Tb_{2,T}\mathrm{Var}[\widetilde{c}_T(u_0,\,k)]<\infty$. *Furthermore, we have* $\widehat{c}_T(u_0,\,k)-c(u_0,\,k)=O_{\mathbb{P}}(\sqrt{Tb_{2,T}})$ *for all* $u_0\in(0,\,1)$.

Proof of Lemma S.A.4. If $Tu_0\notin\mathcal{T}$ then the result follows from Lemma S.A.3. Suppose $Tu_0\in\mathcal{T}$ and $k/Tb_{2,T}\to 0$ (the case $k<0$ is similar and omitted). We omit the subscript j from $A_{j,s-k,T}^0(\omega)$ and from $A_j((s-k)/T,\,\omega)$ since the value j is determined by s and $s-k$, respectively, and can thus be omitted. Using (2.2) we have,

$$\mathbb{E}[\widetilde{c}_T(u_0,\,k)]=\frac{1}{Tb_{2,T}}\sum_{s=k+1}^{T}K_2^*\left(\frac{u_0-(s-k/2)/T}{b_{2,T}}\right)\int_{-\pi}^{\pi}\exp(i\omega k)\,A_{s-k,T}^0(\omega)\,A_{s,T}^0(-\omega)\,d\omega.$$

Since $K_2(x)=0$ for $x<0$, the above sum runs up to $s=Tu_0+k/2T$. Hence, the behavior of $A_{s,T}^0(\omega)$

only matters on a left neighborhood of u_0. Using (2.4) we have,

$$\mathbb{E}\left[\widetilde{c}_T\left(u_0,\, k\right)\right] = \frac{1}{Tb_{2,T}} \sum_{s=k+1}^{T} K_2^*\left(\frac{u_0 - \left(s - k/2\right)/T}{b_{2,T}}\right) \int_{-\pi}^{\pi} \exp\left(i\omega k\right) A\left(\frac{s-k}{T},\, \omega\right) A\left(\frac{s}{T},\, -\omega\right) d\omega + O\left(T^{-1}\right).$$

By the definition of $f\left(\cdot,\, \cdot\right)$, it follows that,

$$\mathbb{E}\left[\widetilde{c}_T\left(u_0,\, k\right)\right] = \frac{1}{Tb_{2,T}} \sum_{s=k+1}^{T} K_2^*\left(\frac{u_0 - \left(s - k/2\right)/T}{b_{2,T}}\right) \int_{-\pi}^{\pi} \exp\left(i\omega k\right) f\left(\frac{s-k/2}{T},\, \omega\right) d\omega + O\left(T^{-1}\right).$$

Let $u_{\epsilon,T} = u_0 - \epsilon_T$, where $\epsilon_T > 0$. Since $f\left(u,\, \omega\right)$ is twice differentiable in u at $u \neq \lambda_j^0$ (cf. Assumption 3.1), by taking a second-order Taylor's expansion of f around $u_{\epsilon,T}$ we have

$$\begin{aligned}
\mathbb{E}\left[\widetilde{c}_T\left(u_0,\, k\right)\right] &= \frac{1}{Tb_{2,T}} \sum_{s=k+1}^{T} K_2^*\left(\frac{u_0 - \left(s - k/2\right)/T}{b_{2,T}}\right) \int_{-\pi}^{\pi} \exp\left(i\omega k\right) f\left(u_{\epsilon,T},\, \omega\right) d\omega \\
&\quad + \frac{1}{Tb_{2,T}} \sum_{s=k+1}^{T} K_2^*\left(\frac{u_0 - \left(s - k/2\right)/T}{b_{2,T}}\right) \int_{-\pi}^{\pi} \exp\left(i\omega k\right) \frac{\partial f\left(u_{\epsilon,T},\, \omega\right)}{\partial u} \left(\frac{s-k/2}{T} - u_{\epsilon,T}\right) d\omega \\
&\quad + \frac{1}{2}\frac{1}{Tb_{2,T}} \sum_{s=k+1}^{T} K_2^*\left(\frac{u_0 - \left(s - k/2\right)/T}{b_{2,T}}\right) \int_{-\pi}^{\pi} \exp\left(i\omega k\right) \frac{\partial^2 f\left(u_{\epsilon,T},\, \omega\right)}{\partial u^2} \left(\frac{s-k/2}{T} - u_{\epsilon,T}\right)^2 d\omega \\
&\quad + o\left(b_{2,T}^2\right) + O\left(\frac{1}{Tb_{2,T}}\right).
\end{aligned}$$

Choose $\epsilon_T = o_{\mathbb{P}}(\max\{b_{2,T}^2,\, 1/(Tb_{2,T})\})$. Using $\int_0^1 K_2\left(x\right) dx = 1$, $K_2\left(x\right) = K_2\left(1-x\right)$ and the definition of $c\left(u_{\epsilon,T},\, k\right)$, the right-hand side above is equal to

$$c\left(u_{\epsilon_T},\, k\right) + \frac{1}{2} b_{2,T}^2 \int_0^1 x^2 K_2\left(x\right) dx \int_{-\pi}^{\pi} \exp\left(i\omega k\right) \frac{\partial^2 f\left(u_{\epsilon_T},\, \omega\right)}{\partial u^2} d\omega + O\left(\frac{1}{Tb_{2,T}}\right) + o\left(b_{2,T}^2\right).$$

Since $c\left(u_0,\, k\right)$ and $\partial^2 f\left(u_0,\, \omega\right)/\partial u^2$ are left-Lipschitz continuous by Assumption 3.1-(iii),

$$c\left(u_{\epsilon_T},\, k\right) - c\left(u_0,\, k\right) = O_{\mathbb{P}}\left(\epsilon_T\right), \qquad \frac{\partial^2 f\left(u_{\epsilon_T},\, \omega\right)}{\partial u^2} - \frac{\partial^2 f\left(u_0,\, \omega\right)}{\partial_- u^2} = O_{\mathbb{P}}\left(\epsilon_T\right).$$

Then,

$$\mathbb{E}\left[\widetilde{c}_T\left(u_0,\, k\right) - c\left(u_0,\, k\right)\right] = \frac{1}{2} b_{2,T}^2 \int_0^1 x^2 K_2\left(x\right) dx \int_{-\pi}^{\pi} \exp\left(i\omega k\right) \frac{\partial^2 f\left(u_0,\, \omega\right)}{\partial_- u^2} d\omega + O\left(\frac{1}{Tb_{2,T}}\right) + o\left(b_{2,T}^2\right).$$

It remains to consider the case $Tu_0 = T\lambda_j^0 \in \mathcal{T}$ and $\left|k\right|/T \to \eta_2 \in (0,\, \lambda_{j+1}^0 - \lambda_j^0)$. Suppose $k < 0$ (the case $k > 0$ is similar and omitted). The derivations for the bias expression are different. Again, using (2.2) we have,

$$\mathbb{E}\left[\widetilde{c}_T\left(u_0,\, k\right)\right] = \frac{1}{Tb_{2,T}} \sum_{s=k+1}^{T} K_2^*\left(\frac{u_0 - \left(s + k/2\right)/T}{b_{2,T}}\right) \int_{-\pi}^{\pi} \exp\left(i\omega k\right) A_{s+k,T}^0\left(\omega\right) A_{s,T}^0\left(-\omega\right) d\omega.$$

Using (2.4), we have

$$\mathbb{E}\left[\widetilde{c}_T\left(u_0,\,k\right)\right]=\frac{1}{Tb_{2,T}}\sum_{s=k+1}^{T}K_2^*\left(\frac{u_0-\left(s+k/2\right)/T}{b_{2,T}}\right)\int_{-\pi}^{\pi}\exp\left(i\omega k\right)A\left(\frac{s+k}{T},\,\omega\right)A\left(\frac{s}{T},\,-\omega\right)d\omega+O\left(T^{-1}\right).$$

We cannot use the property $f_j\left(u,\,\omega\right)=\left|A_j\left(u,\,\omega\right)\right|^2$ for $T_{j-1}^0/T<u=t/T\leq T_j^0/T$ because now $u_0=s+k/2$ implies $s=Tu_0-k/2>Tu_0$. That is, $A_j\left(\left(s+k\right)/T,\,\omega\right)A_{j+1}\left(s/T,\,-\omega\right)$ cannot be approximated by $f_j\left(s-k/2,\,\omega\right)$ for those s such that $s>T_j^0$. However, by taking a second-order Taylor's expansion of A_j about $u_0-\epsilon_{1,T}$ and of A_{j+1} about $v_0+\epsilon_{2,T}$ where $v_0=u_0-k/2T$ and $\epsilon_{1,T},\,\epsilon_{2,T}>0$, we have

$$\begin{aligned}
&\mathbb{E}\left[\widetilde{c}_T\left(u_0,\,k\right)\right]\\
&=\frac{1}{Tb_{2,T}}\sum_{s=k+1}^{T}K_2^*\left(\frac{u_0-\left(s+k/2\right)/T}{b_{2,T}}\right)\int_{-\pi}^{\pi}\exp\left(i\omega k\right)A_{j+1}\left(v_0+\epsilon_{2,T},\,\omega\right)A_j\left(u_0-\epsilon_{1,T},\,-\omega\right)d\omega\\
&\quad+\frac{1}{Tb_{2,T}}\sum_{s=k+1}^{T}K_2^*\left(\frac{u_0-\left(s+k/2\right)/T}{b_{2,T}}\right)\int_{-\pi}^{\pi}\exp\left(i\omega k\right)\left[\frac{\partial A_{j+1}\left(v_0+\epsilon_{2,T},\,\omega\right)}{\partial v}\right.\\
&\quad\times A_j\left(u_0-\epsilon_{1,T},\,-\omega\right)\left(\frac{s}{T}-v_0-\epsilon_{2,T}\right)\\
&\quad\left.+\frac{\partial A_j\left(u_0-\epsilon_{1,T},\,-\omega\right)}{\partial u}A_{j+1}\left(v_0+\epsilon_{2,T},\,\omega\right)\left(\frac{s+k/2}{T}-u_0+\epsilon_{1,T}\right)\right]d\omega\\
&\quad+\frac{1}{2}\frac{1}{Tb_{2,T}}\sum_{s=k+1}^{T}K_2^*\left(\frac{u_0-\left(s+k/2\right)/T}{b_{2,T}}\right)\int_{-\pi}^{\pi}\exp\left(i\omega k\right)\left[\frac{\partial^2 A_{j+1}\left(v_0+\epsilon_{2,T},\,\omega\right)}{\partial v^2}\right.\\
&\quad\times A_j\left(u_0-\epsilon_{1,T},\,-\omega\right)\left(\frac{s}{T}-v_0-\epsilon_{2,T}\right)^2\\
&\quad\left.+\frac{\partial^2 A_j\left(u_0-\epsilon_{1,T},\,-\omega\right)}{\partial u^2}A_{j+1}\left(v_0+\epsilon_{2,T},\,\omega\right)\left(\frac{s+k/2}{T}-u_0+\epsilon_{1,T}\right)^2\right]d\omega\\
&\quad+\frac{1}{Tb_{2,T}}\sum_{s=k+1}^{T}K_2^*\left(\frac{u_0-\left(s+k/2\right)/T}{b_{2,T}}\right)\\
&\quad\times\int_{-\pi}^{\pi}\exp\left(i\omega k\right)\left[\frac{\partial A_{j+1}\left(v_0+\epsilon_{2,T},\,\omega\right)}{\partial v}\frac{\partial A_j\left(u_0-\epsilon_{1,T},\,-\omega\right)}{\partial u}\left(\frac{s}{T}-v_0-\epsilon_{2,T}\right)\left(\frac{s+k/2}{T}-u_0+\epsilon_{1,T}\right)\right]d\omega \qquad \text{(S.8)}\\
&\quad+o\left(b_{2,T}^2\right).
\end{aligned}$$

By Assumption 3.1, $A_j\left(u,\,-\omega\right)$, $\partial A_j\left(u,\,-\omega\right)/\partial u$ and $\partial^2 A_j\left(u,\,-\omega\right)/\partial u^2$ are left-continuous at $u=u_0$, and $A_{j+1}\left(u,\,\omega\right)$, $\partial A_{j+1}\left(u,\,\omega\right)/\partial u$ and $\partial^2 A_{j+1}\left(u,\,\omega\right)/\partial u^2$ are right-continuous at $u=v_0$, thus we have,

$$A_j\left(u_0-\epsilon_{1,T},\,-\omega\right)-A_j\left(u_0,\,-\omega\right)=O_{\mathbb{P}}\left(\epsilon_{1,T}\right),\qquad\frac{\partial A_j\left(u_0-\epsilon_{1,T},\,-\omega\right)}{\partial u}-\frac{\partial A_j\left(u_0,\,-\omega\right)}{\partial_- u}=O_{\mathbb{P}}\left(\epsilon_{1,T}\right),$$

$$\frac{\partial^2 A_j\left(u_0-\epsilon_{1,T},\,-\omega\right)}{\partial u^2}-\frac{\partial^2 A_j\left(u_0,\,-\omega\right)}{\partial_- u^2}=O_{\mathbb{P}}\left(\epsilon_{1,T}\right)$$

$$A_{j+1}\left(v_0+\epsilon_{2,T},\,\omega\right)-A_{j+1}\left(v_0,\,\omega\right)=O_{\mathbb{P}}\left(\epsilon_{2,T}\right),\qquad\frac{\partial A_{j+1}\left(v_0+\epsilon_{2,T},\,\omega\right)}{\partial v}-\frac{\partial A_{j+1}\left(v_0,\,\omega\right)}{\partial_+ v}=O_{\mathbb{P}}\left(\epsilon_{2,T}\right),$$

$$\frac{\partial^2 A_{j+1}\left(v_0+\epsilon_{2,T},\,\omega\right)}{\partial v^2}-\frac{\partial^2 A_{j+1}\left(v_0,\,\omega\right)}{\partial_+ v^2}=O_{\mathbb{P}}\left(\epsilon_{2,T}\right).$$

Choose $\epsilon_{1,T} = o_{\mathbb{P}}(\max\{b_{2,T}^2,\, 1/(Tb_{2,T})\})$ and $\epsilon_{2,T} = o_{\mathbb{P}}(\max\{b_{2,T}^2,\, 1/(Tb_{2,T})\})$. Using the definition of $c(u_0,\, k)$ for $k<0$, (S.8) is equal to,

$$c(u_0,\, k) + b_{2,T}^2 \int_0^1 x^2 K_2(x)\, dx \int_{-\pi}^{\pi} \exp(i\omega k)\left(C_1(u_0,\, \omega) + C_2(u_0,\, \omega) + C_3(u_0,\, \omega)\right) d\omega + O\left(\frac{1}{Tb_{2,T}}\right) + o\left(b_{2,T}^2\right).$$

For $Tu_0 \notin \mathcal{T}$, (S.3) and (S.4) follow by a similar proof as for Lemma S.A.3. Next, let us consider $\mathrm{Var}[\widetilde{c}_T(u_0,\, k)]$ for $p=1$ and $V_{t,T}$ Gaussian. Assume $u_0 = \lambda_j^0$ and $|k|/Tb_{2,T} \to \eta_2 \in (0,\, \lambda_{j+1}^0 - \lambda_j^0)$, we have for $k<0$,

$$\begin{aligned}\mathrm{Var}[\widetilde{c}_T(u_0,\, k)] &= \frac{1}{(Tb_{2,T})^2}\sum_t\sum_s K_2^*\left(\frac{u_0-(t+k/2)/T}{b_{2,T}}\right) K_2^*\left(\frac{u_0-(s+k/2)/T}{b_{2,T}}\right)\mathbb{E}(V_{t,T}V_{s,T})\,\mathbb{E}(V_{t+k,T}V_{s+k,T}) \\ &+ \frac{1}{(Tb_{2,T})^2}\sum_t\sum_s K_2^*\left(\frac{u_0-(t+k/2)/T}{b_{2,T}}\right) K_2^*\left(\frac{u_0-(s+k/2)/T}{b_{2,T}}\right)\mathbb{E}(V_{t,T}V_{s+k,T})\,\mathbb{E}(V_{s,T}V_{t+k,T}).\end{aligned}$$

By (2.4), $A_{s+k,T}^0(\omega)\, A_{t,T}^0(-\omega) = A_j((s+k)/T,\, \omega)\, A_{j+1}(t/T,\, -\omega) + O(T^{-1})$ and $A_{t,T}^0(\omega)\, A_{s,T}^0(-\omega) = A_{j+1}(t/T,\, \omega)\, A_{j+1}(s/T,\, -\omega) + O(T^{-1})$ for $s,\, t = Tu_0 - k/2$. Now take a second order Taylor's expansion of A_{j+1} around $v_0 = u_0 - k/2T + \epsilon_{2,T}$ and of A_j around $u_{\epsilon,T} = u_0 - \epsilon_T$, where $\epsilon_{2,T},\, \epsilon_T > 0$. Applying the manipulations in (S.8) involving A_j and A_{j+1} combined with the same derivations that led to (S.6) we obtain,

$$\begin{aligned}\mathrm{Var}[\widetilde{c}_T(u_0,\, k)] = \int_0^1 K_2(x)^2\, dx \Bigg\{ &\sum_{l=-\infty}^{\infty}[c(v_0,\, l)\, c(u_0,\, l+2k)] \\ &+ \sum_{l=-\infty}^{0}[c(u_0,\, l)\, c(u_0,\, l)] + \sum_{l=1}^{\infty}[c(v_0,\, l)\, c(v_0,\, l)]\Bigg\},\end{aligned} \tag{S.9}$$

where $c(u_0,\, \cdot)$ in the second line above takes the form [cf. the definition of $c(u_0,\, l)$ for $l<0$ at the end of Section 2.1],

$$c(u_0,\, l) = \int_{-\pi}^{\pi} \exp(i\omega l)\, A_j(u_0,\, \omega)\, A_{j+1}(u_0 - l/T,\, \omega)\, d\omega.$$

When $V_{t,T}$ is non-Gaussian, there is an extra term in $\mathrm{Cov}[\widetilde{c}_T^{(j,l)}(u_0,\, k),\, \widetilde{c}_T^{(r,w)}(u_0,\, k)]$, namely

$$\frac{1}{(Tb_{2,T})^2}\sum_t\sum_s K_2^*\left(\frac{u_0-(t+k/2)/T}{b_T}\right) K_2^*\left(\frac{u_0-(s+k/2)/T}{b_T}\right)\kappa_{V,t}^{(j,l,r,w)}(-k,\, s-t,\, s-t-k). \tag{S.10}$$

By Assumption 3.2 with $u = t/T$,

$$\sup_{1\le j\le m_0+1}\ \sup_{\lambda_{j-1}^0 < u \le \lambda_j^0}\left|\kappa_{V,Tu}^{(j,l,r,w)}(-k,\, s-Tu,\, s-Tu-k) - \widetilde{\kappa}_{j,l,r,w}(u,\, -k,\, s-Tu,\, s-Tu-k)\right| = O\left(T^{-1}\right).$$

Taking a second-order Taylor's expansion of $\kappa_{V,Tu}^{(j,l,r,w)}$ with respect to the first argument around $v_0 =$

$u_0 - k/2T + \epsilon_{2,T}$ with $\epsilon_{2,T} > 0$, we have

$$\begin{aligned}
&\frac{1}{(Tb_{2,T})^2} \sum_t \sum_s K_2^* \left(\frac{u_0 - (t + k/2)/T}{b_{2,T}} \right) K_2^* \left(\frac{u_0 - (s + k/2)/T}{b_{2,T}} \right) \kappa_{V,t}^{(j,l,r,w)} (-k,\, s-t,\, s-t-k) \\
&\quad = \frac{1}{(Tb_{2,T})^2} \sum_t \sum_s K_2^* \left(\frac{u_0 - (t + k/2)/T}{b_{2,T}} \right) K_2^* \left(\frac{u_0 - (s + k/2)/T}{b_{2,T}} \right) \\
&\qquad \times \widetilde{\kappa}_{j,l,r,w} (v_0,\, -k,\, s - Tv_0,\, s - Tv_0 - k) \\
&\qquad + \frac{1}{(Tb_{2,T})^2} \sum_t \sum_s K_2^* \left(\frac{u_0 - (t + k/2)/T}{b_{2,T}} \right) K_2^* \left(\frac{u_0 - (s + k/2)/T}{b_{2,T}} \right) \\
&\qquad \times \frac{\partial \widetilde{\kappa}_{j,l,r,w}}{\partial v} (v_0,\, -k,\, s - Tv_0,\, s - Tv_0 - k) (v_0 - t/T) \\
&\qquad + \frac{1}{(Tb_{2,T})^2} \sum_t \sum_s K_2^* \left(\frac{u_0 - (t + k/2)/T}{b_{2,T}} \right) K_2^* \left(\frac{u_0 - (s + k/2)/T}{b_{2,T}} \right) \\
&\qquad \times \frac{\partial^2 \widetilde{\kappa}_{j,l,r,w}}{\partial v^2} (v_0,\, -k,\, s - Tv_0,\, s - Tv_0 - k) (v_0 - t/T)^2 + O\left(T^{-1}\right).
\end{aligned} \tag{S.11}$$

Let $\epsilon_{2,T} = o_{\mathbb{P}}(\max\{b_{2,T}^2,\, 1/(Tb_{2,T})\})$. Since $\widetilde{\kappa}_{j,l,r,w} (v_0,\, \cdot,\, \cdot,\, \cdot)$ is uniformly piecewise Lipschitz continuous by Assumption 3.2-(ii), the first term on the right-hand side above is equal to

$$\begin{aligned}
&\frac{1}{(Tb_{2,T})^2} \sum_t \sum_s K_2^* \left(\frac{u_0 - (t + k/2)/T}{b_T} \right) K_2^* \left(\frac{u_0 - (s + k/2)/T}{b_T} \right) \\
&\qquad \times \left(\widetilde{\kappa}_{j,l,r,w} (v_0,\, -k,\, s - Tv_0,\, s - Tv_0 - k) + O\left(T^{-1}\right) \right).
\end{aligned}$$

The second and third term of (S.11) are of smaller order $o\left(1/Tb_{2,T}\right)$. Thus, (S.10) is equal to

$$\frac{1}{Tb_{2,T}} \int_0^1 K_2^2 (x)\, dx \sum_{h_1=-\infty}^{\infty} \widetilde{\kappa}_{j,l,r,w} (v_0,\, -k,\, h_1,\, h_1 - k) + o\left(\frac{1}{Tb_{2,T}} \right).$$

It remains to derive the expressions for $\mathrm{Var}\left[\widetilde{c}_T (u_0,\, k)\right]$ and $\mathrm{Cov}[\widetilde{c}_T^{(j,l)} (u_0,\, k),\, \widetilde{c}_T^{(r,w)} (u_0,\, k)]$ for the case $|k|/Tb_{2,T} \to 0$. As seen when studying the bias, the behavior of $A^0_{\cdot,T} (\cdot)$ only matters on a left neighborhood of u_0 and thus the result remains the same as in the locally stationary case. The argument involves using first a Taylor's expansion around $u_0 - \epsilon_{1,T}$ with $\epsilon_{1,T} > 0$ and then exploiting left-Lipschitz continuity. As in the proof of Lemma S.A.3, basic manipulations lead to the bound for the MSE. Then, consistency and the rate of convergence follow from the same arguments used there. □

Lemma S.A.5. *Cosnider $p = 1$. Under Assumption 3.1-3.2,* $\sup_{k \geq 1} Tb_{2,T} \mathrm{Var}(\widetilde{\Gamma}(k)) = O(1)$.

Proof of Lemma S.A.5. We have for $k \geq 0$,

$$\mathrm{Var}\left(\widetilde{\Gamma}(k) \right) = \left(\frac{n_T}{T} \right)^2 \sum_{r=0}^{\lfloor T/n_T \rfloor} \sum_{w=0}^{\lfloor T/n_T \rfloor} \mathrm{Cov}\left(\widetilde{c}_T (rn_T/T,\, k),\, \widetilde{c}_T (wn_T/T,\, k) \right),$$

with

$$\begin{aligned}
&\mathrm{Cov}\left(\widetilde{c}_T (rn_T/T,\, k),\, \widetilde{c}_T (wn_T/T,\, k) \right) \\
&\quad = \frac{1}{(Tb_{2,T})^2} \sum_t \sum_s K_2^* \left(\frac{rn_T/T - (t + k/2)/T}{b_T} \right) K_2^* \left(\frac{wn_T/T - (s + k/2)/T}{b_T} \right)
\end{aligned}$$

$$
\begin{aligned}
&\times \mathbb{E}\left(V_{t,T}V_{t+k,T}V_{s,T}V_{s+k,T}\right)\\
&+\frac{1}{\left(Tb_{2,T}\right)^{2}}\sum_{t}\sum_{s}K_{2}^{*}\left(\frac{rn_{T}/T-\left(t+k/2\right)/T}{b_{T}}\right)K_{2}^{*}\left(\frac{wn_{T}/T-\left(s+k/2\right)/T}{b_{T}}\right)\\
&\times \mathbb{E}\left(V_{t,T}V_{s,T}\right)\mathbb{E}\left(V_{t+k,T}V_{s+k,T}\right)\\
&+\frac{1}{\left(Tb_{2,T}\right)^{2}}\sum_{t}\sum_{s}K_{2}^{*}\left(\frac{rn_{T}/T-\left(t+k/2\right)/T}{b_{T}}\right)K_{2}^{*}\left(\frac{wn_{T}/T-\left(s+k/2\right)/T}{b_{T}}\right)\\
&\times \mathbb{E}\left(V_{t,T}V_{t+k,T}\right)\mathbb{E}\left(V_{s,T}V_{s+k,T}\right)\\
&+\frac{1}{\left(Tb_{2,T}\right)^{2}}\sum_{t}\sum_{s}K_{2}^{*}\left(\frac{rn_{T}/T-\left(t+k/2\right)/T}{b_{T}}\right)K_{2}^{*}\left(\frac{wn_{T}/T-\left(s+k/2\right)/T}{b_{T}}\right)\\
&\times \mathbb{E}\left(V_{t,T}V_{s+k,T}\right)\mathbb{E}\left(V_{s,T}V_{t+k,T}\right)-\mathbb{E}\left(\widetilde{c}_{T}\left(rn_{T}/T,\,k\right)\right)\mathbb{E}\left(\widetilde{c}_{T}\left(wn_{T}/T,\,k\right)\right).
\end{aligned}
$$

Proceeding as in the proof of Lemma S.A.4, we have

$$
\begin{aligned}
&\mathrm{Cov}\left(\widetilde{c}_{T}\left(rn_{T}/T,\,k\right),\,\widetilde{c}_{T}\left(wn_{T}/T,\,k\right)\right)\\
&=\frac{1}{Tb_{2,T}}\int_{0}^{1}K_{2}\left(x\right)dx\sum_{h_{1}=-\infty}^{\infty}\widetilde{\kappa}\left(rn_{T}/T,-k,\,h_{1},\,h_{1}-k\right)\\
&\quad+\frac{1}{Tb_{2,T}}\int_{0}^{1}x^{2}K_{2}\left(x\right)dx\sum_{l=-\infty}^{\infty}c\left(rn_{T}/T,\,l\right)c\left(wn_{T}/T,\,l\right)\\
&\quad+\frac{1}{Tb_{2,T}}\int_{0}^{1}x^{2}K_{2}\left(x\right)dx\sum_{l=-\infty}^{\infty}c\left(rn_{T}/T,\,l\right)c\left(wn_{T}/T,\,l+2k\right)+o\left(\frac{1}{Tb_{2,T}}\right),
\end{aligned}
$$

where $\widetilde{\kappa}=\widetilde{\kappa}_{1,1,1,1}$ is the cumulant for the univariate case. Note that

$$
\begin{aligned}
\sum_{l=-\infty}^{\infty}c\left(rn_{T}/T,\,l\right)c\left(wn_{T}/T,\,l+2k\right)&\leq\sum_{l=-\infty}^{\infty}\left|c\left(rn_{T}/T,\,l\right)\right|\sum_{s=-\infty}^{\infty}\left|c\left(wn_{T}/T,\,s+2k\right)\right|\\
&\leq\sum_{l=-\infty}^{\infty}\left|c\left(rn_{T}/T,\,l\right)\right|\sum_{s=-\infty}^{\infty}\left|c\left(wn_{T}/T,\,s\right)\right|.
\end{aligned}
$$

The desired result then follows by Assumption 3.2-(i) and the convergence of approximation to Riemann sums. □

S.A.2 Proofs of the Results of Section 3

S.A.2.1 Proof of Lemma 3.1

It follows by Lemma S.A.4. □

S.A.2.2 Proof of Theorem 3.1

We first prove the result for the locally stationary case (i.e., $m=0$) and then extend it to the general case $m>0$. We begin with the result for the scalar case ($p=1$) and then extend it to the vector case.

Lemma S.A.6. *Suppose* $p=1$, $K_1(\cdot)\in\boldsymbol{K}_1$, $K_2(\cdot)\in\boldsymbol{K}_2$, *Assumption S.A.1-S.A.2 hold,* $b_{1,T},\,b_{2,T}\to0$, $n_T\to\infty$, $n_T/T\to0$ *and* $1/Tb_{1,T}b_{2,T}\to0$. *We have:*

(i) $\lim_{T\to\infty}Tb_{1,T}b_{2,T}\mathrm{Var}(\widetilde{J}_T)=4\pi^2\int K_1^2(y)\,dy\int_0^1K_2^2(x)\,dx(\int_0^1 f(u,\,0)\,du)^2$.

(ii) If $1/Tb_{1,T}^q b_{2,T} \to 0$, $n_T/Tb_{1,T}^q \to 0$ and $b_{2,T}^2/b_{1,T}^q \to 0$ for some $q \in [0, \infty)$ for which $K_{1,q}$, $|\int_0^1 f^{(q)}(u, 0)\, du| \in [0, \infty)$, then $\lim_{T\to\infty} b_{1,T}^{-q}[\mathbb{E}(\widetilde{J}_T - J_T)] = -2\pi K_{1,q} \int_0^1 f^{(q)}(u, 0)\, du$.

(iii) If $n_T/Tb_{1,T}^q \to 0$, $b_{2,T}^2/b_{1,T}^q \to 0$ and $Tb_{1,T}^{2q} b_{2,T} \to \gamma \in (0, \infty)$ for some $q \in [0, \infty)$ for which $K_{1,q}$, $|\int_0^1 f^{(q)}(u, 0)\, du| \in [0, \infty)$, then

$$\begin{aligned}
&\lim_{T\to\infty} \mathrm{MSE}\left(Tb_{1,T} b_{2,T},\, \widehat{J}_T,\, 1\right) \\
&\quad = 4\pi^2 \left[\gamma K_{1,q}^2 \left(\int_0^1 f^{(q)}(u, 0)\, du\right)^2 + \int K_1^2(y)\, dy \int K_2^2(x)\, dx \left(\int_0^1 f(u, 0)\, du\right)^2\right].
\end{aligned}$$

Proof of Lemma S.A.6. We begin with part (i). Note that for any fixed non-negative $\tau_1, \tau_2 \in \mathbb{R}$,

$$\begin{aligned}
&\mathrm{Cov}\left(V_s V_{s-\tau_1},\, V_l V_{l-\tau_2}\right) \\
&\quad = \mathbb{E}\left[\left(V_s V_{s-\tau_1} - \mathbb{E}\left(V_s V_{s-\tau_1}\right)\right)\left(V_l V_{l-\tau_2} - \mathbb{E}\left(V_l V_{l-\tau_2}\right)\right)\right] \\
&\quad = \mathbb{E}\left(V_s V_{s-\tau_1} V_l V_{l-\tau_2}\right) - \Gamma_{s/T}(\tau_1)\Gamma_{l/T}(\tau_2) - \Gamma_{s/T}(\tau_1)\Gamma_{l/T}(\tau_2) - \Gamma_{l/T}(l-s)\Gamma_{(l-\tau_2)/T}(l-s-\tau_2+\tau_1) \\
&\qquad - \Gamma_{(l-\tau_2)/T}(l-s-\tau_2)\Gamma_{l/T}(l-s+\tau_1) + \Gamma_{s/T}(\tau_1)\Gamma_{l/T}(\tau_2) + \Gamma_{l/T}(l-s)\Gamma_{(l-\tau_2)/T}(l-s-\tau_2+\tau_1) \\
&\qquad + \Gamma_{(l-\tau_2)/T}(l-s-\tau_2)\Gamma_{l/T}(l-s+\tau_1) \\
&\quad = \kappa_{V,s}(-\tau_1,\, l-s,\, l-s-\tau_2) + \Gamma_{s/T}(s-l)\Gamma_{(l-\tau_2)/T}(l-s-\tau_2+\tau_1) \\
&\qquad + \Gamma_{s/T}(s-l+\tau_2)\Gamma_{l/T}(l-s+\tau_1) \\
&\quad = \kappa_{V,s}(-\tau_1,\, l-s,\, l-s-\tau_2) + \Gamma_{l/T}(l-s)\Gamma_{(l-\tau_2)/T}(l-s-\tau_2+\tau_1) \\
&\qquad + \Gamma_{(l-\tau_2)/T}(l-s-\tau_2)\Gamma_{l/T}(l-s+\tau_1).
\end{aligned}$$

For large T, we have by Lemma S.A.2: $\Gamma_{(l-\tau_2)/T}(k) - \Gamma_{l/T}(k) = O(\tau_2/T)$, and $\Gamma_{(s-\tau_1)/T}(k) = \Gamma_{s/T}(k) + O(\tau_1/T)$ uniformly in k, l and s. Apply the changes in variables $w = s - l$ and $v = l$, then

$$\begin{aligned}
&\sum_{s=\tau_1+1}^{T} \sum_{l=\tau_2+1}^{T} \mathrm{Cov}\left(V_s V_{s-\tau_1},\, V_l V_{l-\tau_2}\right) \\
&\quad = \sum_{s=\tau_1+1}^{T} \sum_{l=\tau_2+1}^{T} \kappa_{V,s}(-\tau_1,\, l-s,\, l-s-\tau_2) \\
&\qquad + \sum_{v=\tau_2+1}^{T} \sum_{w=\tau_1+1-v}^{T-v} \left[\Gamma_{v/T}(-w)\Gamma_{v/T}(-w+\tau_2-\tau_1) + \Gamma_{v/T}(-w-\tau_2)\Gamma_{v/T}(-w+\tau_1)\right] \\
&\qquad + \sum_{v=\tau_2+1}^{T} \sum_{w=\tau_1+1-v}^{T-v} \left[\Gamma_{v/T}(-w)\, O(\tau_2/T) + O(\tau_2/T)\,\Gamma_{v/T}(-w+\tau_1)\right].
\end{aligned} \tag{S.12}$$

A bound for the term involving $\Gamma_{v/T}(-w)\, O(\tau_2/T)$ in (S.12) is

$$\begin{aligned}
\sum_{v=\tau_2+1}^{T} \sum_{w=\tau_1+1-v}^{T-v} \left|\Gamma_{v/T}(-w)\right| O(\tau_2/T) &\leq O(\tau_2/T) \sum_{v=\tau_2+1}^{T} \sum_{w=\tau_1+1-v}^{T-v} \sup_{(v/T)\in[0,\,1]} \left|\Gamma_{v/T}(w)\right| \\
&\leq O\left(T^{-1}\right),
\end{aligned} \tag{S.13}$$

where we have used Assumption S.A.2-(i). The argument for the term involving $O(\tau_2/T)\,\Gamma_{v/T}(-w+\tau_1)$ is analogous. We next evaluate the covariance of $\widetilde{c}_T(t/T,\, k)$. For any $1 \leq t_1, t_2 \leq T$ and (without loss of generality) non-negative integers $\tau_1, \tau_2 \in \mathbb{R}$, apply the following changes in variables $w = s - l$ and $v = l$, so that

$$
\begin{aligned}
&Tb_{2,T}\mathrm{Cov}\left[\widetilde{c}_T\left(t_1/T,\,\tau_1\right),\,\widetilde{c}_T\left(t_2/T,\,\tau_2\right)\right]\\
&\quad= Tb_{2,T}\left(\frac{1}{Tb_{2,T}}\right)^2\sum_{s=\tau_1+1}^{T}\sum_{l=\tau_2+1}^{T}\\
&\qquad\times K_2^*\left(\frac{\left(t_1-\left(s-\tau_1/2\right)\right)/T}{b_{2,T}}\right)K_2^*\left(\frac{\left(t_2-\left(l-\tau_2/2\right)\right)/T}{b_{2,T}}\right)\mathrm{Cov}\left(V_sV_{s-\tau_1},\,V_lV_{l-\tau_2}\right)\\
&\quad=\frac{1}{Tb_{2,T}}\sum_{v=\tau_2+1}^{T}\sum_{w=\tau_1+1-v}^{T-v}K_2^*\left(\frac{\left(t_1-\left(v+w-\tau_1/2\right)\right)/T}{b_{2,T}}\right)K_2^*\left(\frac{\left(t_2-\left(v-\tau_2/2\right)\right)/T}{b_{2,T}}\right)\\
&\qquad\times\left\{\left[\Gamma_{v/T}\left(-w\right)\Gamma_{v/T}\left(-w+\tau_2-\tau_1\right)+\Gamma_{v/T}\left(-w-\tau_2\right)\Gamma_{v/T}\left(-w+\tau_1\right)\right]\right\}\\
&\qquad+\frac{1}{Tb_{2,T}}\sum_{s=\tau_1+1}^{T}\sum_{l=\tau_2+1}^{T}K_2^*\left(\frac{\left(t_1-\left(s-\tau_1/2\right)\right)/T}{b_{2,T}}\right)\\
&\qquad\times K_2^*\left(\frac{\left(t_2-\left(v-\tau_2/2\right)\right)/T}{b_{2,T}}\right)\kappa_{V,s}\left(-\tau_1,\,l-s,\,l-s-\tau_2\right)+A_T,
\end{aligned}
$$

where

$$
\begin{aligned}
A_T\triangleq{}&\frac{1}{Tb_{2,T}}\sum_{v=\tau_2+1}^{T}\sum_{w=\tau_1+1-v}^{T-v}K_2^*\left(\frac{\left(t_1-\left(v+w-\tau_1/2\right)\right)/T}{b_{2,T}}\right)K_2^*\left(\frac{\left(t_2-\left(v-\tau_2/2\right)\right)/T}{b_{2,T}}\right)\\
&\times\left\{\left[\Gamma_{v/T}\left(w\right)O\left(\tau_2/T\right)+O\left(\tau_2/T\right)\Gamma_{v/T}\left(w+\tau_1\right)\right]\right\}.
\end{aligned}
$$

Using (S.13), we have $A_T=o\left(T^{-1}\right)$. Then, using the change of variable $z=v/Tb_{2,T}$,

$$
\begin{aligned}
&Tb_{2,T}\mathrm{Cov}\left[\widetilde{c}_T\left(t_1/T,\,\tau_1\right),\,\widetilde{c}_T\left(t_2/T,\,\tau_2\right)\right]\\
&\quad=\frac{1}{Tb_{2,T}}\sum_{v=\tau_2+1}^{T}\sum_{w=\tau_1+1-v}^{T-v}K_2^*\left(\frac{\left(t_1-v-w+\tau_1/2+v-v\right)/T}{b_{2,T}}\right)K_2^*\left(\frac{\left(t_2-v+\tau_2/2\right)/T}{b_{2,T}}\right)\\
&\qquad\times\left\{\left[\Gamma_v\left(-w\right)\Gamma_v\left(-w+\tau_2-\tau_1\right)+\Gamma_v\left(-w-\tau_2\right)\Gamma_v\left(-w+\tau_1\right)\right]\right\}\\
&\qquad+\frac{1}{Tb_{2,T}}\sum_{s=\tau_1+1}^{T}\sum_{l=\tau_2+1}^{T}K_2^*\left(\frac{\left(t_1-\left(s-\tau_1/2\right)\right)/T}{b_{2,T}}\right)K_2^*\left(\frac{\left(t_2-\left(v-\tau_2/2\right)\right)/T}{b_{2,T}}\right)\\
&\qquad\times\kappa_{V,s}\left(-\tau_1,\,l-s,\,l-s-\tau_2\right)+A_T\\
&\quad=\frac{1}{Tb_{2,T}}\sum_{z=(\tau_2+1)/Tb_{2,T}}^{1/b_{2,T}}\sum_{w=\tau_1+1-Tb_{2,T}z}^{T-Tb_{2,T}z}K_2^*\left(\frac{\left(t_1+w+\tau_1/2\right)/T}{b_{2,T}}-z\right)K_2^*\left(\frac{\left(t_2+\tau_2/2\right)/T}{b_{2,T}}-z\right)\\
&\qquad\times\left\{\left[\Gamma_{zTb_{2,T}}\left(-w\right)\Gamma_{zTb_{2,T}}\left(-w+\tau_2-\tau_1\right)+\Gamma_{zTb_{2,T}}\left(-w-\tau_2\right)\Gamma_{zTb_{2,T}}\left(-w+\tau_1\right)\right]\right\}\\
&\qquad+\frac{1}{Tb_{2,T}}\sum_{s=\tau_1+1}^{T}\sum_{l=\tau_2+1}^{T}K_2^*\left(\frac{\left(t_1-\left(s+\tau_1/2\right)\right)/T}{b_{2,T}}\right)K_2^*\left(\frac{\left(t_2-\left(v+\tau_2/2\right)\right)/T}{b_{2,T}}\right)\\
&\qquad\times\kappa_{V,s}\left(-\tau_1,\,l-s,\,l-s-\tau_2\right)+A_T.
\end{aligned}
\tag{S.14}
$$

Thus, with $u = t_1/T$ and $v = t_2/T$, the limit of the first term of (S.14) is equal to

$$\int_0^1 K_2^2(x)\,dx \left\{ \sum_{w=-\infty}^{\infty} \left[\Gamma_u(w)\,\Gamma_v(-w+\tau_2-\tau_1) + \Gamma_u(w+\tau_2)\,\Gamma_v(-w+\tau_1) \right] \right\}.$$

When $\tau_1 = \tau_2 = k$ and $t = t_1 = t_2$, we have

$$\begin{aligned} Tb_{2,T}\mathrm{Var}\left(\widetilde{c}_T(t/T,\,k)\right) &= \int_0^1 K_2(x)^2\,dx \left\{ \sum_{w=-\infty}^{\infty} \left[\Gamma_u(w)\,\Gamma_u(w) + \Gamma_u(w+k)\,\Gamma_u(w-k) \right] \right\} \\ &= \int_0^1 K_2(x)^2\,dx \left\{ \sum_{h=-\infty}^{\infty} \left[\Gamma_u(h)\,\Gamma_u(h) + \Gamma_u(h+2k)\,\Gamma_u(h) \right] \right\}, \end{aligned}$$

where $u = t/T$ and we have used the change in variable $h = w - k$. Next, we consider $\mathrm{Cov}[\widetilde{\Gamma}(\tau_1),\,\widetilde{\Gamma}(\tau_2)]$. Note that,

$$\begin{aligned} &Tb_{2,T}\mathrm{Cov}\left[\widetilde{\Gamma}(\tau_1),\,\widetilde{\Gamma}(\tau_2)\right] \\ &\quad\to \int_0^1 K_2^2(x)\,dx \int_0^1\int_0^1 \left\{ \sum_{h=-\infty}^{\infty} \left[\Gamma_u(h)\,\Gamma_u(h-\tau_2+\tau_1) + \Gamma_v(-h-\tau_2)\,\Gamma_v(-h-\tau_1) \right] \right\} dvdu. \end{aligned}$$

The latter can be used to evaluate $\mathrm{Var}[\sum_{k=-T+1}^{T-1} K_1(b_{1,T}k)\,\widetilde{\Gamma}(k)]$ as follows,

$$\begin{aligned} &Tb_{1,T}b_{2,T}\mathrm{Var}\left[\sum_{k=-T+1}^{T-1} K_1(b_{1,T}k)\,\widetilde{\Gamma}(k)\right] \\ &\quad= 2b_{1,T}\sum_{k=-T+1}^{T-1}\sum_{j=0}^{T-1} K_1(b_{1,T}k)\,K_1(b_{1,T}j) \qquad\qquad \text{(S.15)} \\ &\quad\times \left(\frac{n_T}{T}\right)^2 \sum_{r=0}^{T/n_T}\sum_{b=0}^{T/n_T} \frac{1}{Tb_{2,T}} \sum_{s=k+1}^{T}\sum_{l=j+1}^{T} K_2^*\left(\frac{(rn_T+1)-(s+k/2)}{Tb_{2,T}}\right) K_2^*\left(\frac{(bn_T+1)-(l+j/2)}{Tb_{2,T}}\right) \\ &\quad\times \Big(\Big[\Gamma_{l/T}(l-s)\,\Gamma_{l/T}(l-s-j+k) + \Gamma_{l/T}(-s+l-\tau_2)\,\Gamma_{l/T}(-s+l+k) \Big] \\ &\quad+ \kappa_{V,s}(-k,\,l-s,\,l-s-j) \Big) + o(1), \end{aligned}$$

where the $o(1)$ term follows from $A_T = o(b_{1,T}/T)$. The term involving $\kappa_{V,s}(-k,\,l-s,\,l-s-j)$ is dominated by

$$Cb_{1,T}\left| \sum_{k=-\infty}^{\infty}\sum_{j=0}^{\infty}\sum_{w=-\infty}^{\infty} \sup_s \kappa_{V,s}(-k,\,-w,\,-w-j) \right| = O(b_{1,T}),$$

where $C < \infty$ and we have used Assumption S.A.2-(i). Now let $w = s - l$ and $v = l$ and rewrite (S.15) as

$$Tb_{1,T}b_{2,T}\mathrm{Var}\left[\sum_{k=-T+1}^{T-1} K_1(b_{1,T}k)\,\widetilde{\Gamma}(k)\right]$$

$$= 2b_{1,T} \sum_{k=-T+1}^{T-1} \sum_{j=0}^{T-1} K_1 (b_{1,T}k) K_1 (b_{1,T}j)$$
$$\times \left(\frac{n_T}{T}\right)^2 \sum_{r=0}^{T/n_T} \sum_{b=0}^{T/n_T} \frac{1}{Tb_{2,T}} \sum_{v=j+1}^{T} \sum_{w=k+1-v}^{T-v}$$
$$\times K_2^* \left(\frac{(rn_T+1)-(w+v-k/2)}{Tb_{2,T}}\right) K_2^* \left(\frac{(bn_T+1)-(v-j/2)}{Tb_{2,T}}\right)$$
$$\times \left[\Gamma_{v/T}(-w)\Gamma_{v/T}(-w+j-k)+\Gamma_{v/T}(-w-j)\Gamma_{v/T}(-w+k)\right] + o(1) + O(b_{1,T}).$$

We next show that the term involving $\Gamma_{v/T}(-w-j)\Gamma_{v/T}(-w+k)$ vanishes in the limit. Using a change in variables $z_1 = j+k$ and $z = w+j$, the latter is bounded by

$$4b_{1,T} \sum_{j=0}^{T-1} \sum_{z_1=j}^{T-1+j} K_1 (b_{1,T}(z_1-j)) K_1 (b_{1,T}j) \left(\frac{n_T}{T}\right)^2 \sum_{r=0}^{T/n_T} \sum_{b=0}^{T/n_T}$$
$$\times \frac{1}{Tb_{2,T}} \sum_{v=j+1}^{T} \sum_{z=(z_1-j)+1-v+j}^{T-v+j} K_2^* \left(\frac{(rn_T+1)-(z-j+v-(z_1-j)/2)}{Tb_{2,T}}\right)$$
$$\times K_2^* \left(\frac{((bn_T+1)-(v-j/2))/T}{b_{2,T}}\right) \left[\Gamma_{v/T}(-z)\Gamma_{v/T}(-z+z_1)\right]. \quad \text{(S.16)}$$

Making the change in variable $z_2 = jb_{1,T}$, (S.16) can be expressed as,

$$4b_{1,T} \sum_{z_2=0}^{(T-1)/b_{1,T}} \sum_{z_1=z_2/b_{1,T}}^{T-1+z_2/b_{1,T}} K_1 (b_{1,T}(z_1-z_2/b_{1,T})) K_1 (z_2) \left(\frac{n_T}{T}\right)^2 \sum_{r=0}^{T/n_T} \sum_{b=0}^{T/n_T}$$
$$\times \frac{1}{Tb_{2,T}} \sum_{v=z_2/b_{1,T}+1}^{T} \sum_{z=z_1+1-v}^{T-v+z_2/b_{1,T}} K_2^* \left(\frac{(rn_T+1)-(z-z_2/b_{1,T}+v-(z_1-z_2/b_{1,T})/2)}{Tb_{2,T}}\right)$$
$$\times K_2^* \left(\frac{((bn_T+1)-(v-z_2/2b_{1,T}))/T}{b_{2,T}}\right) \left[\Gamma_{v/T}(-z)\Gamma_{v/T}(-z+z_1)\right],$$

which converges to zero because the range of summation over z_1 tends to infinity.

Next, let us consider the term of (S.15) involving $\Gamma_{v/T}(-w)\Gamma_{v/T}(-w+j-k)$. With the changes in variables $u_1 = k-j$ and $u_2 = j$, this term becomes

$$b_{1,T} \sum_{u_2=-T+1}^{T-1} \sum_{u_1=-u_2-T+1}^{T-1-u_2} K_1 (b_{1,T}(u_2+u_1)) K_1 (b_{1,T}u_2) \left(\frac{n_T}{T}\right)^2 \sum_{r=0}^{T/n_T} \sum_{b=0}^{T/n_T} \frac{1}{Tb_{2,T}} \sum_{v=u_2+1}^{T} \sum_{w=u_2+u_1+1-v}^{T-v} \quad \text{(S.17)}$$
$$\times K_2^* \left(\frac{(rn_T+1)-(w+v-(u_1+u_2)/2)}{Tb_{2,T}}\right) K_2^* \left(\frac{(bn_T+1)-(v-u_2/2)}{Tb_{2,T}}\right)$$
$$\times \left[\Gamma_{v/T}(w)\Gamma_{v/T}(-w-u_1)\right].$$

Apply the change in variable $z = b_{1,T}u_2$ and consider the lattice points $z_n = nb_{1,T}$, where $n = -T, \ldots, T$. As $T \to \infty$, the distance between the lattice points $z_n = nb_{1,T}$ converges to zero and the highest lattice point converges to infinity. Hence, (S.17) can be expressed as,

$$\sum_{z_n=-(T-1)b_{1,T}}^{(T-1)b_{1,T}} \sum_{u_1=-z_n/b_{1,T}-T+1}^{T-1-z_n/b_{1,T}} K_1\left(b_{1,T}u_1+z_n\right)K_1\left(z_n\right) \tag{S.18}$$
$$\times\left(\frac{n_T}{T}\right)^2 \sum_{r=0}^{T/n_T}\sum_{b=0}^{T/n_T}\frac{1}{Tb_{2,T}}$$
$$\times\sum_{v=z_n/b_{1,T}+1}^{T}\sum_{w=z_n/b_{1,T}+u_1+1-v}^{T-v} K_2\left(\frac{\left(\left(rn_T+1\right)-\left(w+v-\left(z_n/b_{1,T}+u_1\right)/2\right)\right)/T}{b_{2,T}}\right)$$
$$\times K_2\left(\frac{\left(\left(bn_T+1\right)-\left(v-z/2b_{1,T}\right)\right)/T}{b_{2,T}}\right)\left[\Gamma_{v/T}\left(w\right)\Gamma_{v/T}\left(-w-u_1\right)\right].$$

By Lemma S.A.1, $\Gamma_{v/T}(w)\Gamma_{v/T}(-w-u_1)=c(v/T,\,-w)\,c(v/T,\,w+u_1)+O(T^{-1})$. By taking a second order Taylor's expansion of $c(v/T,\,-w)$ around rn_T/T and of $c((v-u_1/1)/T,\,w+u_1/2)$ around bn_T/T, we have

$$\sum_{v=z_n/b_{1,T}+1}^{T}\sum_{w=z_n/b_{1,T}+u_1+1-v}^{T-v} K_2\left(\frac{\left(\left(rn_T+1\right)-\left(w+v-\left(z_n/b_{1,T}+u_1\right)/2\right)\right)/T}{b_{2,T}}\right)$$
$$\times K_2\left(\frac{\left(\left(bn_T+1\right)-\left(v-z/2b_{1,T}\right)\right)/T}{b_{2,T}}\right)\left[c\left(v/T,\,-w\right)c\left(v/T,\,w+u_1\right)\right]$$
$$=\int_0^1 K_2(x)^2\,dx\,c\left(rn_T/T,\,-w\right)c\left(bn_T/T,\,w+u_1\right)$$
$$+b_{2,T}^2\int_0^1 x^2K_2(x)^2\,dx\frac{\partial}{\partial v}c\left(v,\,-w\right)|_{v=rn_T/T}\frac{\partial}{\partial v}c\left(v,\,w+u_1\right)|_{v=bn_T/T}$$
$$+2^{-1}b_{2,T}^2\int_0^1 x^2K_2(x)^2\,dx\frac{\partial^2}{\partial v^2}c\left(v,\,-w\right)|_{v=rn_T/T}c\left(bn_T/T,\,w+u_1\right)$$
$$+2^{-1}b_{2,T}^2\int_0^1 x^2K_2(x)^2\,dx\,c\left(rn_T/T,\,-w\right)\frac{\partial^2}{\partial v^2}c\left(v,\,w+u_1\right)|_{v=bn_T/T}$$
$$+o\left(b_{2,T}^2\right)+O\left(\frac{1}{Tb_{2,T}}\right).$$

We can now use Lemma S.A.1 backward to show that the limit of (S.18) is equal to

$$\int K_1(y)^2\,dy\int_0^1 K_2(x)^2\,dx\int_0^1\int_0^1\sum_{u_1=\infty}^{\infty}\sum_{w=-\infty}^{\infty}\left[\Gamma_u(w)\Gamma_a(w+u_1)\right]du da$$
$$=4\pi^2\int K_1(y)^2\,dy\int_0^1 K_2(x)^2\,dx\left(\int_0^1 f(u,\,0)\,du\right)\left(\int_0^1 f(a,\,0)\,da\right).$$

This proves the result of part (i). We now move to part (ii). Let

$$J_{c,T}\triangleq\int_0^1 c(u,\,0)+2\sum_{k=1}^{T-1}\int_0^1 c(u,\,k)\,du.$$

We begin with the following relationship,

$$\mathbb{E}\left(\widetilde{J}_T - J_T\right) = \sum_{k=-T+1}^{T-1} K_1\left(b_{1,T}k\right)\mathbb{E}\left(\widetilde{\Gamma}\left(k\right)\right) - J_{c,T} + \left(J_{c,T} - J_T\right).$$

Using Lemma S.A.3, we have for any $-T+1 \leq k \leq T-1$,

$$\begin{aligned}
&\mathbb{E}\left(\frac{n_T}{T}\sum_{r=0}^{T/n_T}\widetilde{c}_T\left(rn_T/T,\, k\right) - \int_0^1 c\left(u,\, k\right)du\right) \\
&= \frac{n_T}{T}\sum_{r=0}^{T/n_T}\left(c\left(rn_T/T,\, k\right) + \frac{1}{2}b_{2,T}^2\int_0^1 x^2K_2\left(x\right)dx\frac{\partial^2}{\partial^2 u}c\left(u,\, k\right)|_{u=rn_T/T} + o\left(b_{2,T}^2\right) + O\left(\frac{1}{b_{2,T}T}\right)\right) \\
&\quad - \int_0^1 c\left(u,\, k\right)du \\
&= O\left(\frac{n_T}{T}\right) + \frac{1}{2}b_{2,T}^2\int_0^1 x^2K_2\left(x\right)dx\int_0^1\frac{\partial^2}{\partial^2 u}c\left(u,\, k\right)du + o\left(b_{2,T}^2\right) + O\left(\frac{1}{Tb_{2,T}}\right),
\end{aligned}$$

where the last equality follows from the convergence of approximations to Riemann sums. This leads to,

$$\begin{aligned}
&b_{1,T}^{-q}\mathbb{E}\left(\widetilde{J}_T - J_{c,T}\right) \\
&\quad = -b_{1,T}^{-q}\sum_{k=-T+1}^{T}\left(1 - K_1\left(b_{1,T}k\right)\right)\int_0^1 c\left(u,\, k\right)du \\
&\qquad + \frac{1}{2}\frac{b_{2,T}^2}{b_{1,T}^q}\int_0^1 x^2K_2\left(x\right)dx\sum_{k=-T+1}^{T}K_1\left(b_{1,T}k\right)\int_0^1\frac{\partial^2}{\partial^2 u}c\left(u,\, k\right)du + O\left(\frac{1}{Tb_{1,T}^q b_{2,T}}\right) + O\left(\frac{n_T}{Tb_{1,T}^q}\right) \\
&\quad = -b_{1,T}^{-q}\sum_{k=-T+1}^{T}\left(1 - K_1\left(b_{1,T}k\right)\right)\int_0^1 c\left(u,\, k\right)du \\
&\qquad - \frac{1}{2}b_{2,T}^2\int_0^1 x^2K_2\left(x\right)dxO\left(1\right) + \frac{1}{2}\frac{b_{2,T}^2}{b_{1,T}^q}\int_0^1 x^2K_2\left(x\right)dxO\left(1\right) + O\left(\frac{1}{Tb_{1,T}^q b_{2,T}}\right) + O\left(\frac{n_T}{Tb_{1,T}^q}\right),
\end{aligned}$$

since $|\sum_{k=-\infty}^{\infty}|k|^q\int_0^1\left(\partial^2/\partial^2 u\right)c\left(u,\, k\right)du| < \infty$ by Assumption S.A.2-(i). Since $J_{c,T} - J_T = O\left(T^{-1}\right)$, we conclude that

$$\lim_{T\to\infty} b_{1,T}^{-q}\mathbb{E}\left(\widetilde{J}_T - J_T\right) = -2\pi K_{1,q}\int_0^1 f^{(q)}\left(u,\, 0\right)du,$$

because $b_{2,T}^2/b_{1,T}^q \to 0$. It remains to show part (iii). Note that $Tb_{1,T}b_{2,T} = Tb_{1,T}b_{2,T}b_{1,T}^{2q}/b_{1,T}^{2q} = b_{1,T}^{-2q}/(1/Tb_{1,T}^{2q+1}b_{2,T}) = b_{1,T}^{-2q}/\left(1/\left(\gamma + o\left(1\right)\right)\right)$. Hence, using part (i)-(ii), we deduce the desired result, namely,

$$\begin{aligned}
&\lim_{T\to\infty}\text{MSE}\left(Tb_{1,T}b_{2,T},\, \widetilde{J}_T,\, 1\right) \\
&\qquad = \lim_{T\to\infty} b_{1,T}^{-2q}\mathbb{E}\left[\left(\widetilde{J}_T - J_T\right)^2\right]\left(\gamma + o\left(1\right)\right) + \lim_{T\to\infty} Tb_{1,T}b_{2,T}\text{Var}\left(\widetilde{J}_T\right) \\
&\qquad = 4\pi^2\left[\gamma K_{1,q}^2\left(\int_0^1 f^{(q)}\left(u,\, 0\right)du\right)^2 + \int K_1^2\left(y\right)dy\int_0^1 K_2^2\left(x\right)dx\left(\int_0^1 f\left(u,\, 0\right)du\right)^2\right]. \qquad \square
\end{aligned}$$

Lemma S.A.7. *Suppose $K_1(\cdot) \in \boldsymbol{K}_1$, $K_2(\cdot) \in \boldsymbol{K}_2$, Assumption S.A.1-S.A.2 hold, $b_{1,T}$, $b_{2,T} \to 0$, $n_T \to \infty$, $n_T/T \to 0$ and $1/Tb_{1,T}b_{2,T} \to 0$. Then, part (i)-(iii) of Theorem 3.1 hold.*

Proof of Lemma S.A.7. We begin with part (i). We provide the expression for the asymptotic covariance between the (a, l) and (m, n) elements of $\widetilde{J}_T$:

$$\begin{aligned}
&Tb_{1,T}b_{2,T}\text{Cov}\left[\sum_{k=-T+1}^{T-1} K_1(b_{1,T}k)\,\widetilde{\Gamma}^{(a,l)}(k),\ \sum_{j=-T+1}^{T-1} K_1(b_{1,T}j)\,\widetilde{\Gamma}^{(m,n)}(j)\right] \\
&= 4b_{1,T}\sum_{k=0}^{T-1}\sum_{j=0}^{T-1} K_1(b_{1,T}k)\,K_1(b_{1,T}j)\left(\frac{n_T}{T}\right)^2 \sum_{r=0}^{T/n_T}\sum_{b=0}^{T/n_T}\frac{1}{Tb_{2,T}}\sum_{s=k+1}^{T}\sum_{h=j+1}^{T} \qquad \text{(S.19)}\\
&\times K_2^*\left(\frac{((rn_T+1)-(s-k/2))/T}{b_{2,T}}\right)K_2^*\left(\frac{((bn_T+1)-(h-j/2))/T}{b_{2,T}}\right) \\
&\times\Big\{\kappa_{V,s}^{(a,l,m,n)}(-k,\, h-s,\, h-s-j) \\
&+\left[\Gamma_{h/T}^{(a,m)}(h-s)\,\Gamma_{h/T}^{(l,n)}(h-s-j+k)+\Gamma_{h/T}^{(a,n)}(h-s-j)\,\Gamma_{h/T}^{(l,m)}(h-s+k)\right]\Big\}+o(1),
\end{aligned}$$

where the $o(1)$ term follows from using (S.13). The term involving $\kappa_{V,s}^{(a,l,m,n)}(-k,\, h-s,\, h-s-j)$ is negligible as for the scalar case. The limit of the term involving $\Gamma_{h/T}^{(a,m)}(h-s)\,\Gamma_{h/T}^{(l,n)}(h-s+j-k)$ is, according to the derivations to prove part (i) of Lemma S.A.6,

$$4\pi^2\int K_1(y)^2\,dy\int_0^1 K_2(x)^2\,dx\left(\int_0^1 f^{(a,m)}(u,\,0)\,du\right)\left(\int_0^1 f^{(l,n)}(v,\,0)\,dv\right). \qquad \text{(S.20)}$$

Similarly, the limit of the term involving $\Gamma_{h/T}^{(a,n)}(s-h-j)\,\Gamma_{h/T}^{(l,m)}(s-h+k)$ is the same as (S.20) but with m and n interchanged. The commutation-tensor product formula arises from the fact that the asymptotic covariances between $\widetilde{J}_T^{(a,l)}$ and $\widetilde{J}_T^{(m,n)}$ for $a,\, l,\, m,\, n \le p$ are of the same form as the covariances between X_aX_l and X_mX_n, where $X = (X_1, \ldots, X_p)' \sim \mathscr{N}(0,\,\Sigma)$. The formula then follows from $\text{Var}(\text{vec}(XX')) = \text{Var}(X\otimes X) = (I + C_{pp})\,\Sigma\otimes\Sigma$. The proof of part (ii) of the lemma follows that of the scalar case with minor changes. Since part (iii) simply uses part (i)-(ii), it follows that

$$\begin{aligned}
&\lim_{T\to\infty}\text{MSE}\left(Tb_{1,T}b_{2,T},\,\widetilde{J}_T,\,W\right) \\
&\quad=\lim_{T\to\infty}\gamma b_{1,T}^{-2q}\mathbb{E}\left(\widetilde{J}_T-J_T\right)'W\mathbb{E}\left(\widetilde{J}_T-J_T\right)+\lim_{T\to\infty}Tb_{1,T}b_{2,T}\,\text{tr}W\,\text{Var}\left(\text{vec}\left(\widetilde{J}_T\right)\right),
\end{aligned}$$

converges to the desired limit. □

Lemma S.A.8. *Suppose $p = 1$, $K_1(\cdot) \in \boldsymbol{K}_1$, $K_2(\cdot) \in \boldsymbol{K}_2$, Assumption 3.1-3.2 hold, $b_{1,T}$, $b_{2,T} \to 0$, $n_T \to \infty$, $n_T/T \to 0$ and $1/Tb_{1,T}b_{2,T} \to 0$. Then, (i)-(iii) of Lemma S.A.6 continue to hold.*

Proof of Lemma S.A.8. We assume without loss of generality that $m_0 = 1$ and provide the proof only for the single break case. Hence, the break date is T_2^0 (i.e., $T_1^0 = 0$ and $T_3^0 = T$). Note that by standard properties of approximations to Riemann sums, $\overline{\Gamma}(k) \to \int_0^1 (c(u,\,k))\,du$ even when $c(\cdot,\,k)$ has a finite number of discontinuities in u, where

$$\overline{\Gamma}(k) \triangleq \frac{n_T}{T-n_T}\sum_{r=0}^{\lfloor (T-n_T)/n_T\rfloor} c(rn_T/T,\,k).$$

Since the results in Lemma S.A.4 about the order of the bias and variance of $\widetilde{c}_T(u_0, k)$ are the same to their counterpart results in Lemma S.A.3, the proof of Lemma S.A.6 can be repeated with the following changes. We begin with part (i). For any fixed non-negative $\tau_1, \tau_2 \in \mathbb{R}$,

$$
\begin{aligned}
&\mathrm{Cov}\left(V_s V_{s-\tau_1}, V_l V_{l-\tau_2}\right) \\
&\quad= \kappa_{V,s}\left(-\tau_1, l-s, l-s-\tau_2\right) + \Gamma_{l/T}\left(l-s\right)\Gamma_{(l-\tau_2)/T}\left(l-s-\tau_2+\tau_1\right) \\
&\qquad+ \Gamma_{(l-\tau_2)/T}\left(l-s-\tau_2\right)\Gamma_{l/T}\left(l-s+\tau_1\right).
\end{aligned}
$$

When $l = T_2^0$ and $\tau_2 < 0$, Lemma S.A.2 cannot be applied because of the discontinuity in the spectrum of $\{V_{t,T}\}$ at time $t = T_2^0$. Thus, the relation $\Gamma_{(l-\tau_2)/T}(k) - \Gamma_{l/T}(k) = (\tau_2/T)$ for $l = T_2^0$ and $\tau_2 < 0$ does not hold. One has to carry $\Gamma_{(l-\tau_2)/T}(k)$ through the proof. Applying the changes in variables $w = s - l$ and $v = l$, we have

$$
\begin{aligned}
&\sum_{s=\tau_1+1}^{T}\sum_{l=\tau_2+1}^{T}\mathrm{Cov}\left(V_{s/T}V_{(s-\tau_1)/T}, V_{l/T}V_{(l-\tau_2)/T}\right) \\
&\quad= \sum_{s=\tau_1+1}^{T}\sum_{l=\tau_2+1}^{T}\kappa_{V,s}\left(-\tau_1, l-s, l-s-\tau_2\right) \\
&\qquad+ \sum_{v=\tau_2+1}^{T}\sum_{w=\tau_1+1-v}^{T-\tau_2-v}\Big[\Gamma_{v/T}\left(-w\right)\Gamma_{(v-\tau_2)/T}\left(-w+\tau_2-\tau_1\right) + \Gamma_{(v-\tau_2)/T}\left(-w-\tau_2\right)\Gamma_{v/T}\left(-w+\tau_1\right)\Big].
\end{aligned}
\tag{S.21}
$$

We next evaluate the covariance of $\widetilde{c}_T(t/T, k)$. For any $1 \leq t_1, t_2 \leq T$ and (without loss of generality) non-negative integers $\tau_1, \tau_2 \in \mathbb{R}$,

$$
\begin{aligned}
&Tb_{2,T}\mathrm{Cov}\left[\widetilde{c}_T\left(t_1/T, \tau_1\right), \widetilde{c}_T\left(t_2/T, \tau_2\right)\right] \\
&\quad= Tb_{2,T}\left(\frac{1}{Tb_{2,T}}\right)^2\sum_{s=\tau_1+1}^{T}\sum_{v=\tau_2+1}^{T} \\
&\qquad\times K_2^*\left(\frac{\left(t_1-\left(s-\tau_1/2\right)\right)/T}{b_{2,T}}\right)K_2^*\left(\frac{\left(t_2-\left(v-\tau_2/2\right)\right)/T}{b_{2,T}}\right)\mathrm{Cov}\left(V_sV_{s-\tau_1}, V_lV_{l-\tau_2}\right) \\
&\quad= \frac{1}{Tb_{2,T}}\sum_{v=\tau_2+1}^{T}\sum_{w=\tau_1+1-v}^{T-v}K_2^*\left(\frac{\left(t_1-\left(v+w-\tau_1/2\right)\right)/T}{b_{2,T}}\right)K_2^*\left(\frac{\left(t_2-\left(v-\tau_2/2\right)\right)/T}{b_{2,T}}\right) \\
&\qquad\times\left\{\Big[\Gamma_{v/T}\left(-w\right)\Gamma_{(v-\tau_2)/T}\left(-w+\tau_2-\tau_1\right) + \Gamma_{(v-\tau_2)/T}\left(-w-\tau_2\right)\Gamma_{v/T}\left(-w+\tau_1\right)\Big]\right\} \\
&\qquad+ \frac{1}{Tb_{2,T}}\sum_{s=\tau_1+1}^{T}\sum_{l=\tau_2+1}^{T}K_2^*\left(\frac{\left(t_1-\left(s-\tau_1/2\right)\right)/T}{b_{2,T}}\right) \\
&\qquad\times K_2^*\left(\frac{\left(t_2-\left(v-\tau_2/2\right)\right)/T}{b_{2,T}}\right)\kappa_{V,s}\left(-\tau_1, l-s, l-s-\tau_2\right).
\end{aligned}
$$

Then, using the change of variable $z = v/Tb_{2,T}$,

$$
\begin{aligned}
&Tb_{2,T}\mathrm{Cov}\left[\widetilde{c}_T\left(t_1/T, \tau_1\right), \widetilde{c}_T\left(t_2/T, \tau_2\right)\right] \\
&\quad= \frac{1}{Tb_{2,T}}\sum_{v=\tau_2+1}^{T}K_2^*\left(\frac{\left(t_1-v-w-\tau_1/2+v-v\right)/T}{b_{2,T}}\right)K_2^*\left(\frac{\left(t_2-zTb_{2,T}-\tau_2/2\right)/T}{b_{2,T}}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\times\left\{\left[\Gamma_{zb_{2,T}}(-w)\,\Gamma_{zb_{2,T}-\tau_2/T}(-w+\tau_2-\tau_1)+\Gamma_{zb_{2,T}-\tau_2/T}(-w-\tau_2)\,\Gamma_{zb_{2,T}}(-w+\tau_1)\right]\right\}\\
&+\frac{1}{Tb_{2,T}}\sum_{s=\tau_1+1}^{T}\sum_{l=\tau_2+1}^{T}K_2^*\left(\frac{(t_1-(s-\tau_1/2))/T}{b_{2,T}}\right)\\
&\times K_2^*\left(\frac{(t_2-(v+\tau_2/2))/T}{b_{2,T}}\right)\kappa_{V,s}(-\tau_1,\,l-s,\,l-s-\tau_2)\\
=&\,\frac{1}{Tb_{2,T}}\sum_{z=(\tau_2+1)/Tb_{2,T}}^{1/b_{2,T}}\sum_{w=\tau_1+1-zTb_{2,T}}^{T-z/Tb_{2,T}}K_2^*\left(\frac{(t_1+w-\tau_1/2)/T}{b_{2,T}}-z\right)K_2^*\left(\frac{(t_2-\tau_2/2)/T}{b_{2,T}}-z\right)\\
&\times\left\{\left[\Gamma_{zb_{2,T}}(-w)\,\Gamma_{zb_{2,T}-\tau_2/T}(-w+\tau_2-\tau_1)+\Gamma_{zb_{2,T}-\tau_2/T}(-w-\tau_2)\,\Gamma_{zb_{2,T}}(-w+\tau_1)\right]\right\}\\
&+\frac{1}{Tb_{2,T}}\sum_{s=\tau_1+1}^{T}\sum_{l=\tau_2+1}^{T}K_2^*\left(\frac{(t_1-(s-\tau_1/2))/T}{b_{2,T}}\right)\\
&\times K_2^*\left(\frac{(t_2-(v+\tau_2/2))/T}{b_{2,T}}\right)\kappa_{V,s}(-\tau_1,\,l-s,\,l-s-\tau_2)\,.
\end{aligned}
\tag{S.22}
$$

By Lemma S.A.1, $\Gamma_{zb_{2,T}}(-w)\,\Gamma_{zb_{2,T}-\tau_2/T}(-w+\tau_2-\tau_1)=c(zb_{2,T}/T,\,w)\,c(zb_{2,T}-\tau_2/T,\,w-\tau_2+\tau_1)+O(T^{-1})$. We need to distinguish two cases. The first case involves both t_1 and t_2 being continuity points (i.e., $t_1,\,t_2\neq T_2^0$). The second case involves either t_1 or t_2 (or both) being discontinuity points (i.e., $t_1=T_2^0$ or $t_2=T_2^0$, or $t_1=t_2=T_2^0$). The first case is the one considered in Lemma S.A.6 and thus we omit the details. For the second case, we cannot apply the same argument as in Lemma S.A.6. Suppose $t_1=T_2^0$ whereas $t_2\neq T_2^0$. Let $u_{1,\epsilon,T}=t_1/T-\epsilon_{1,T}$, $\epsilon_{1,T}>0$. We proceed as in (S.8) by taking a second order Taylor's expansion of $c(zb_{2,T}/T,\,w)$ around $u_{1,\epsilon,T}$ and then use the left-Lipschitz continuity at t_1/T. Repeat this argument for $c(zb_{2,T}-\tau_2/T,\,w+\tau_2)$. For $c(zb_{2,T}-\tau_2/T,\,w-\tau_2+\tau_1)$ and $c(zb_{2,T}/T,\,w-\tau_1)$, take a Taylor's expansion around t_2/T. Finally, use Lemma S.A.1 backward to obtain

$$
c(t_1/T,\,w)\,c(t_2,\,w-\tau_2+\tau_1)=\Gamma_{t_1/T}(-w)\,\Gamma_{t_2/T}(-w+\tau_2-\tau_1)+O\left(T^{-1}\right).
$$

Thus, with $u=t_1/T$ and $v=t_2/T$, the limit of the first term of (S.22) is equal to

$$
\int_0^1K_2^2(x)\,dx\left\{\sum_{w=-\infty}^{\infty}\left[\Gamma_u(w)\,\Gamma_v(-w+\tau_2-\tau_1)+\Gamma_u(w+\tau_2)\,\Gamma_v(-w+\tau_1)\right]\right\}. \tag{S.23}
$$

For the sub-case where only t_2 is a discontinuity point, use a Taylor's expansion of $c(zb_{2,T}/T,\,w)$ and $c(zb_{2,T}-\tau_2/T,\,w+\tau_2)$ around t_1/T, and proceed as in (S.8) by taking a second order Taylor's expansion of $c(zb_{2,T}-\tau_2/T,\,w-\tau_2+\tau_1)$ and $c(zb_{2,T}/T,\,w-\tau_1)$ around $u_{2,\epsilon,T}=t_2/T-\epsilon_{2,T}$, $\epsilon_{2,T}>0$ and then use the left-Lipschitz continuity at t_2/T. Again using Lemma S.A.1 backward leads to (S.23). For the final case where $t_1=t_2=T_2^0$ we need to proceed as in the previous two sub-cases with $t_1=T_2^0$ and $t_2=T_2^0$ being discontinuity points. This would lead to (S.23). We can use (S.23) to obtain,

$$
\begin{aligned}
&Tb_{2,T}\mathrm{Cov}\left[\widetilde{\Gamma}(\tau_1)\,,\,\widetilde{\Gamma}(\tau_2)\right]\\
&\quad\to\int_0^1K_2^2(x)\,dx\int_0^1\int_0^1\left\{\sum_{h=-\infty}^{\infty}\left[\Gamma_u(h)\,\Gamma_u(h-\tau_2+\tau_1)+\Gamma_v(-h-\tau_2)\,\Gamma_v(-h-\tau_1)\right]\right\}dvdu.
\end{aligned}
$$

In (S.22) the term involving $\kappa_{V,s}(-\tau_1,\,l-s,\,l-s-\tau_2)$ is negligible as in Lemma S.A.6 while the term involving $\Gamma_{(l-\tau_2)/T}(-w-j)\,\Gamma_{l/T}(-w+k)$ vanishes in the limit using the same argument as in the proof of Lemma S.A.6. This proves the result of part (i).

We move to part (ii). Let

$$J_{c,T}=\int_0^1 c(u,\,0)\,du+2\sum_{k=1}^{T-1}\int_0^1 c(u,\,k)\,du,$$

and $\mathcal{T}_C \triangleq \{\{0,\, n_T,\dots,\, T-n_T,\, T\}/\mathcal{T}\}$. We begin with the following relationship,

$$\mathbb{E}\left(\widetilde{J}_T-J_T\right)=\sum_{k=-T+1}^{T-1}K_1(b_{1,T}k)\,\mathbb{E}\left(\widetilde{\Gamma}(k)\right)-J_{c,T}+(J_{c,T}-J_T)\,.$$

Using Lemma S.A.4, we have for any $-T+1\leq k\leq T-1$,

$$\begin{aligned}
&\mathbb{E}\left(\frac{n_T}{T}\sum_{r=0}^{T/n_T}\widetilde{c}_T(rn_T/T,\,k)-\int_0^1 c(u,\,k)\,du\right)\\
&=\frac{n_T}{T}\sum_{r=0}^{T/n_T}c(rn_T/T,\,k)-\int_0^1 c(u,\,k)\,du\\
&\quad+\frac{1}{2}b_{2,T}^2\int_0^1 x^2K_2(x)\,dx\int_0^1\frac{\partial^2}{\partial^2u}c(u,\,k)\,du+o\left(b_{2,T}^2\right)+O\left(\frac{1}{Tb_{2,T}}\right)\\
&\quad+\frac{1}{2}b_{2,T}^2\int_0^1 x^2K_2(x)\,dx\\
&\quad\times\int_0^1\left(\int_{-\pi}^{\pi}\exp(i\omega k)\left(C_1(u,\,\omega)+C_2(u,\,\omega)+C_3(u,\,\omega)\right)d\omega\mathbf{1}\{Tu\in\mathcal{T}\}\right)du\\
&\quad+o\left(b_{2,T}^2\right)+O\left(\frac{1}{Tb_{2,T}}\right)\\
&=O\left(\frac{n_T}{T}\right)+\frac{1}{2}b_{2,T}^2\int_0^1 x^2K_2(x)\,dx\int_0^1\frac{\partial^2}{\partial^2u}c(u,\,k)\,du+o\left(b_{2,T}^2\right)+O\left(\frac{1}{Tb_{2,T}}\right),
\end{aligned}$$

where the last equality follows from the convergence of approximations to Riemann sums and from the fact that $\mathbf{1}\{Tu\in\mathcal{T}\}$ has zero Lebesgue measure. Thus, $b_{1,T}^{-q}\mathbb{E}(\widetilde{J}_T-J_{c,T})$ has the same form as in the locally stationary case. The relation $J_{c,T}-J_T=O(T^{-1})$ continues to hold for SLS processes in virtue of Lemma S.A.1. Hence, $\lim_{T\to\infty}b_{1,T}^{-q}\mathbb{E}(\widetilde{J}_T-J_T)=-2\pi K_{1,q}\int_0^1 f^{(q)}(u,\,0)\,du$. Part (iii) follows from part (i)-(ii). $\square$

Proof of Theorem 3.1. We can now complete the proof of Theorem 3.1. We begin with part (i). We provide the expression for the asymptotic covariance between the $(a,\,l)$ and $(m,\,n)$ elements of $\widetilde{J}_T$:

$$\begin{aligned}
&Tb_{1,T}b_{2,T}\mathrm{Cov}\left[\sum_{k=-T+1}^{T-1}K_1(b_{1,T}k)\,\widetilde{\Gamma}^{(a,l)}(k)\,,\;\sum_{j=-T+1}^{T-1}K_1(b_{1,T}j)\,\widetilde{\Gamma}^{(m,n)}(j)\right]\\
&=b_{1,T}\sum_{k=-T+1}^{T-1}\sum_{j=-T+1}^{T-1}K_1(b_{1,T}k)\,K_1(b_{1,T}j)\left(\frac{n_T}{T}\right)^2\sum_{r=0}^{T/n_T}\sum_{b=0}^{T/n_T}\frac{1}{Tb_{2,T}}\sum_{s=k+1}^{T}\sum_{h=j+1}^{T}
\end{aligned}\qquad\text{(S.24)}$$

$$
\times K_2^* \left(\frac{((rn_T+1)-(s-k/2))/T}{b_{2,T}} \right) K_2^* \left(\frac{((bn_T+1)-(h-j/2))/T}{b_{2,T}} \right)
$$
$$
\times \left\{ \kappa_{V,s}^{(a,l,m,n)} (-k,\, h-s,\, h-s-j) \right.
$$
$$
\left. + \left[\Gamma_{h/T}^{(a,m)} (h-s)\, \Gamma_{(h-j)/T}^{(l,n)} (h-s-j+k) + \Gamma_{(h-j)/T}^{(a,n)} (h-s-j)\, \Gamma_{h/T}^{(l,m)} (h-s+k) \right] \right\}.
$$

As for the scalar case, the term involving $\kappa_{V,s}^{(a,l,m,n)} (-k,\, h-s,\, h-s-j)$ is negligible. The limit of the term involving $\Gamma_{h/T}^{(a,m)} (h-s)\, \Gamma_{(h-j)/T}^{(l,n)} (h-s-j+k)$ is, according to the derivations for the proof of part (i) of Lemma S.A.8,

$$
4\pi^2 \int K_1 (y)^2 \, dy \int_0^1 K_2 (x)^2 \, dx \left(\int_0^1 f^{(a,m)} (u,\, 0)\, du \right) \left(\int_0^1 f^{(l,n)} (v,\, 0)\, dv \right). \tag{S.25}
$$

Similarly, the limit of the term involving $\Gamma_{(h-j)/T}^{(a,n)} (s-h-j)\, \Gamma_{h/T}^{(l,m)} (s-h+k)$ is the same as (S.25) but with m and n interchanged. The commutation-tensor product formula follows from the same argument as in Lemma S.A.7. The proof of part (ii) of the theorem follows from that of the scalar case with minor changes. Since part (iii) simply uses part (i)-(ii), it follows that

$$
\lim_{T\to\infty} \mathrm{MSE} \left(Tb_{1,T} b_{2,T},\, \widetilde{J}_T,\, W \right)
$$
$$
= \lim_{T\to\infty} \gamma b_{1,T}^{-2q} \mathbb{E} \left(\widetilde{J}_T - J_T \right)' W \mathbb{E} \left(\widetilde{J}_T - J_T \right) + \lim_{T\to\infty} Tb_{1,T} b_{2,T} \operatorname{tr} W \operatorname{Var} \left(\operatorname{vec} \left(\widetilde{J}_T \right) \right),
$$

converges to the desired limit. □

S.A.2.3 Proof of Theorem 3.2

Under Assumption 3.2, $|| \int_0^1 f^{(0)} (u,\, 0) || < \infty$. In view of $K_{1,0} = 0$, Theorem 3.1-(i,ii) [with $q = 0$ in part (ii)] implies $\widetilde{J}_T - J_T = o_{\mathbb{P}} (1)$. Noting that $\widehat{J}_T - \widetilde{J}_T = o_{\mathbb{P}} (1)$ if and only if $b'\widehat{J}_T b - b'\widetilde{J}_T b = o_{\mathbb{P}} (1)$ for arbitrary $b \in \mathbb{R}^p$ we shall provide the proof only for the scalar case. We first show that $\sqrt{T} b_{1,T} (\widehat{J}_T - \widetilde{J}_T) = O_{\mathbb{P}} (1)$ under Assumption 3.3. Let $\widetilde{J}_T (\beta)$ denote the estimator that uses $\{V_{t,T} (\beta)\}$. A mean-value expansion of $\widetilde{J}_T(\widehat{\beta}) (= \widehat{J}_T)$ about β_0 yields

$$
\sqrt{T} b_{1,T} (\widehat{J}_T - \widetilde{J}_T) = b_{1,T} \frac{\partial}{\partial \beta'} \widetilde{J}_T (\bar{\beta}) \sqrt{T} (\widehat{\beta} - \beta_0)
$$
$$
= b_{1,T} \sum_{k=-T+1}^{T-1} K_1 (b_{1,T} k) \frac{\partial}{\partial \beta'} \widehat{\Gamma} (k) |_{\beta=\bar{\beta}} \sqrt{T} (\widehat{\beta} - \beta_0), \tag{S.26}
$$

for some $\bar{\beta}$ on the line segment joining $\widehat{\beta}$ and β_0. Note also that $\widehat{c} (rn_T/T,\, k)$ depends on β although we omit it. We have for $k \geq 0$ (the case $k < 0$ is similar and omitted),

$$
\left\| \frac{\partial}{\partial \beta'} \widehat{c} (rn_T/T,\, k) \right\| |_{\beta=\bar{\beta}} \tag{S.27}
$$
$$
= \left\| (Tb_{2,T})^{-1} \sum_{s=k+1}^{T} K_2^* \left(\frac{(r+1)\, n_T - (s-k/2)}{Tb_{2,T}} \right) \right.
$$
$$
\left. \times \left(V_s (\beta) \frac{\partial}{\partial \beta'} V_{s-k} (\beta) + \frac{\partial}{\partial \beta'} V_s (\beta)\, V_{s-k} (\beta) \right) \right\| |_{\beta=\bar{\beta}}
$$

$$
\leq 2\left(\left(Tb_{2,T}\right)^{-1}\sum_{s=1}^{T}K_{2}^{*}\left(\frac{\left(r+1\right)n_{T}-\left(s-k/2\right)}{Tb_{2,T}}\right)^{2}\sup_{s\geq1}\sup_{\beta\in\Theta}\left(V_{s}\left(\beta\right)\right)^{2}\right)^{1/2}
$$

$$
\times\left(\left(Tb_{2,T}\right)^{-1}\sum_{s=1}^{T}K_{2}^{*}\left(\frac{\left(r+1\right)n_{T}-\left(s-k/2\right)}{Tb_{2,T}}\right)^{2}\sup_{s\geq1}\sup_{\beta\in\Theta}\left\Vert \frac{\partial}{\partial\beta'}V_{s}\left(\beta\right)\right\Vert ^{2}\right)^{1/2}
$$

$$
=O_{\mathbb{P}}\left(1\right),
$$

where we have used the boundedness of the kernel K_2 (and thus of K_2^*), Assumption 3.3-(ii,iii) and Markov's inequality to each term in parentheses; also $\sup_{s\geq1}\mathbb{E}\sup_{\beta\in\Theta}\left\Vert V_{s}\left(\beta\right)\right\Vert ^{2}<\infty$ under Assumption 3.3-(ii,iii) by a mean-value expansion and,

$$
\left(Tb_{2,T}\right)^{-1}\sum_{s=k+1}^{T}K_{2}^{*}\left(\left(\left(r+1\right)n_{T}-\left(s+k/2\right)\right)/Tb_{2,T}\right)^{2}\rightarrow\int_{0}^{1}K_{2}^{2}\left(x\right)dx<\infty.
$$

Then, (S.26) becomes

$$
b_{1,T}\sum_{k=T+1}^{T-1}K_{1}\left(b_{1,T}k\right)\frac{\partial}{\partial\beta'}\widehat{\Gamma}\left(k\right)|_{\beta=\bar{\beta}}\sqrt{T}\left(\widehat{\beta}-\beta_{0}\right)
$$

$$
\leq b_{1,T}\sum_{k=-T+1}^{T-1}\left|K_{1}\left(b_{1,T}k\right)\right|\frac{n_{T}}{T}\sum_{r=0}^{T/n_{T}}O_{\mathbb{P}}\left(1\right)O_{\mathbb{P}}\left(1\right)
$$

$$
=O_{\mathbb{P}}\left(1\right),
$$

where the last equality uses $b_{1,T}\sum_{k=-T+1}^{T-1}\left|K_{1}\left(b_{1,T}k\right)\right|\rightarrow\int\left|K_{1}\left(x\right)\right|dx<\infty$. This concludes the proof of part (i) of Theorem 3.2 because $\sqrt{T}b_{1,T}\rightarrow\infty$ by assumption.

The next step is to show that $\sqrt{Tb_{1,T}}(\widehat{J}_{T}-\widetilde{J}_{T})=o_{\mathbb{P}}\left(1\right)$ under the assumptions of Theorem 3.2-(ii). A second-order Taylor's expansion gives

$$
\sqrt{Tb_{1,T}}\left(\widehat{J}_{T}-\widetilde{J}_{T}\right)=\left[\sqrt{b_{1,T}}\frac{\partial}{\partial\beta'}\widetilde{J}_{T}\left(\beta_{0}\right)\right]\sqrt{T}\left(\widehat{\beta}-\beta_{0}\right)
$$

$$
+\frac{1}{2}\sqrt{T}\left(\widehat{\beta}-\beta_{0}\right)'\left[\sqrt{b_{1,T}}\frac{\partial^{2}}{\partial\beta\partial\beta'}\widetilde{J}_{T}\left(\overline{\beta}\right)/\sqrt{T}\right]\sqrt{T}\left(\widehat{\beta}-\beta_{0}\right)
$$

$$
\triangleq G_{T}'\sqrt{T}\left(\widehat{\beta}-\beta_{0}\right)+\frac{1}{2}\sqrt{T}\left(\widehat{\beta}-\beta_{0}\right)'H_{T}\sqrt{T}\left(\widehat{\beta}-\beta_{0}\right).
$$

Proceeding as in (S.27) but now using Assumption 3.4-(ii),

$$
\left\Vert \frac{\partial^{2}}{\partial\beta\partial\beta'}\widehat{c}\left(rn_{T}/T,\,k\right)\right\Vert \Big|_{\beta=\bar{\beta}}
$$

$$
=\left\Vert \left(Tb_{2,T}\right)^{-1}\sum_{s=k+1}^{T}K_{2}^{*}\left(\frac{\left(\left(r+1\right)n_{T}-\left(s+k/2\right)\right)/T}{b_{2,T}}\right)\left(\frac{\partial^{2}}{\partial\beta\partial\beta'}V_{s}\left(\beta\right)V_{s-k}\left(\beta\right)\right)\right\Vert \Big|_{\beta=\bar{\beta}}
$$

$$
=O_{\mathbb{P}}\left(1\right),
$$

and thus,

$$\begin{aligned}\|H_T\| &\leq \left(\frac{b_{1,T}}{T}\right)^{1/2} \sum_{k=-T+1}^{T-1} |K_1(b_{1,T}k)| \sup_{\beta\in\Theta} \left\|\frac{\partial^2}{\partial\beta\partial\beta'}\widehat{\Gamma}(k)\right\| \\ &\leq \left(\frac{b_{1,T}}{T}\right)^{1/2} \sum_{k=-T+1}^{T-1} |K_1(b_{1,T}k)| O_{\mathbb{P}}(1) \\ &\leq \left(\frac{1}{Tb_{1,T}}\right)^{1/2} b_{1,T} \sum_{k=-T+1}^{T-1} |K_1(b_{1,T}k)| O_{\mathbb{P}}(1) = o_{\mathbb{P}}(1),\end{aligned}$$

since $Tb_{1,T} \to \infty$. Next, we want to show that $G_T = o_{\mathbb{P}}(1)$. We apply the results of Theorem 3.1-(i,ii) to $\widetilde{J}_T$ where the latter is constructed using $(V_t',\, \partial V_t/\partial\beta' - \mathbb{E}(\partial V_t/\partial\beta'))'$ rather than just with V_t. The first row and column of the off-diagonal elements of $\widetilde{J}_T$ are now (written as column vectors)

$$\begin{aligned}A_1 &\triangleq \sum_{k=-T+1}^{T-1} K_1(b_{1,T}k) \frac{n_T}{T} \sum_{r=0}^{T/n_T} \frac{1}{Tb_{2,T}} \\ &\times \sum_{s=k+1}^{T} K_2^*\left(\frac{((r+1)n_T - (s+k/2))/T}{b_{2,T}}\right) V_s \left(\frac{\partial}{\partial\beta}V_{s-k} - \mathbb{E}\left(\frac{\partial}{\partial\beta}V_s\right)\right) \\ A_2 &\triangleq \sum_{k=-T+1}^{T-1} K_1(b_{1,T}k) \frac{n_T}{T} \sum_{r=0}^{T/n_T} \frac{1}{Tb_{2,T}} \\ &\times \sum_{s=k+1}^{T} K_2^*\left(\frac{((r+1)n_T - (s+k/2))/T}{b_{2,T}}\right) \left(\frac{\partial}{\partial\beta}V_s - \mathbb{E}\left(\frac{\partial}{\partial\beta}V_s\right)\right) V_{s-k}.\end{aligned}$$

By Theorem 3.1-(i,ii), each expression above is $O_{\mathbb{P}}(1)$. Since

$$\begin{aligned}G_T &= \sqrt{b_{1,T}}(A_1 + A_2) + \sqrt{b_{1,T}} \sum_{k=-T+1}^{T-1} K_1(b_{1,T}k) \frac{n_T}{T} \sum_{r=0}^{T/n_T} \frac{1}{Tb_{2,T}} \\ &\times \sum_{s=k+1}^{T} K_2^*\left(\frac{((r+1)n_T - (s+k/2))/T}{b_{2,T}}\right) (V_s + V_{s-k}) \mathbb{E}\left(\frac{\partial}{\partial\beta}V_s\right) \\ &\leq \sqrt{b_{1,T}}(A_1 + A_2) + A_3 \sup_{s\leq T}\left|\mathbb{E}\left(\frac{\partial}{\partial\beta}V_s\right)\right|,\end{aligned}$$

where

$$\begin{aligned}A_3 &= \sqrt{b_{1,T}} \sum_{k=-T+1}^{T-1} |K_1(b_{1,T}k)| \frac{n_T}{T} \sum_{r=0}^{T/n_T} \frac{1}{Tb_{2,T}} \\ &\times \sum_{s=k+1}^{T} \left|K_2^*\left(\frac{((r+1)n_T - (s-k/2))/T}{b_{2,T}}\right)\right| |(V_s + V_{s-k})|,\end{aligned}$$

it remains to show that A_3 is $o_{\mathbb{P}}(1)$. Note that

$$\mathbb{E}\left(A_3^2\right) \leq b_{1,T} \sum_{k=-T+1}^{T-1} \sum_{j=-T+1}^{T-1} |K_1(b_{1,T}k) K_1(b_{1,T}j)| \, 4 \left(\frac{n_T}{T}\right)^2 \sum_{r=0}^{T/n_T} \sum_{b=0}^{T/n_T}$$

$$\times \frac{1}{Tb_{2,T}} \frac{1}{Tb_{2,T}} \sum_{s=1}^{T} \sum_{l=1}^{T} K_2^* \left(\frac{((r+1)\, n_T - (s-k/2))\, /T}{b_{2,T}} \right)$$
$$\times K_2^* \left(\frac{((b+1)\, n_T - (l-j/2))\, /T}{b_{2,T}} \right) |\mathbb{E}\,(V_s V_l)|\,,$$

and that $\mathbb{E}\,(V_s V_l) = c\,(u,\, h) + O\,(T^{-1})$ where $h = s-l$ and $u = s/T$ by Lemma S.A.1. Since $\sum_{h=-\infty}^{\infty} \sup_{u\in[0,\,1]} |c\,(u,\, h)| < \infty$, we have

$$\mathbb{E}\left(A_3^2\right) \leq \frac{1}{Tb_{1,T}b_{2,T}} \left(b_{1,T} \sum_{k=-T+1}^{T-1} |K_1\,(b_{1,T}k)| \right)^2 \int_0^1 K_2^2\,(x)\, dx \int_0^1 \sum_{h=-\infty}^{\infty} |c\,(u,\, h)|\, du = o\,(1)\,.$$

This implies $G_T = o_{\mathbb{P}}\,(1)$. It follows that $\sqrt{Tb_{1,T}}(\widehat{J}_T - \widetilde{J}_T) = o_{\mathbb{P}}\,(1)$ which concludes the proof of part (ii) because $\sqrt{Tb_{1,T}b_{2,T}}(\widetilde{J}_T - J_T) = O_{\mathbb{P}}\,(1)$ by Theorem 3.1-(iii).

Finally, we need to consider part (iii). Let

$$\xi_T \triangleq Tb_{1,T} \left(\text{vec}\left(\widehat{J}_T - J_T\right)' W \text{vec}\left(\widehat{J}_T - J_T\right) - \text{vec}\left(\widetilde{J}_T - J_T\right)' W \text{vec}\left(\widetilde{J}_T, - J_T\right) \right).$$

By part (ii), we know that $\sqrt{Tb_{1,T}}(\widehat{J}_T - J_T) = O_{\mathbb{P}}\,(1)$ and $\sqrt{Tb_{1,T}}(\widehat{J}_T - \widetilde{J}_T) = o_{\mathbb{P}}\,(1)$. This implies

$$Tb_{1,T} \left(\text{vec}\left(\widehat{J}_T - J_T\right)' W_T \text{vec}\left(\widehat{J}_T - J_T\right) - \text{vec}\left(\widetilde{J}_T - J_T\right)' W_T \text{vec}\left(\widetilde{J}_T, - J_T\right) \right) \xrightarrow{\mathbb{P}} 0.$$

Then, using Assumption 3.5, $\xi_T = o_{\mathbb{P}}\,(1)$ and since $|\xi_T|$ is bounded we have $\mathbb{E}\,(\xi_T) \to 0$ by Lemma A1 in Andrews (1991). □

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