Supplement Not for Online Publication to "Theory of Evolutionary Spectra for Heteroskedasticity and Autocorrelation Robust Inference in Possibly Misspecified and Nonstationary Models"

Alessandro Casini*

University of Rome Tor Vergata

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Abstract

This supplemental material is not for publication and is structured as follows. Section N.A reviews how to apply the proposed DK-HAC estimator in GMM and IV contexts. Section N.B contains the proofs of the results of Section 2 and 4-5.

N.A Implementation of DK-HAC in GMM and IV Models

Section N.A.1 reviews the DK-HAC estimation in GMM models while Section N.A.2 considers IV models.

N.A.1 GMM

We begin with the GMM setup [cf. Hansen (1982)]. For a k-vector β_* of unknown parameters, we have the moment condition $\mathbb{E}m_t(\beta_*) = 0$ where $m_t(\beta)$ is a p-vector of functions of the data and parameters where $p \geq k$. The GMM estimator $\widehat{\beta}$ is defined as the solution to $\min_\beta m_T(\beta)' \widehat{W}_{2,T} m_T(\beta)$, where $m_T(\beta) = T^{-1}\sum_{t=1}^T m_t(\beta)$ is the sample average of the vector of sample moments $m_t(\beta)$ and $\widehat{W}_{2,T}$ is a (possibly) random, symmetric weighting matrix. The asymptotic covariance matrix of $\widehat{\beta}$ is given by $H = \lim_{T\to\infty} H_T$ where

$$H_T = \left(L_T' W_{2,T} L_T\right)^{-1} L_T' W_{2,T} J_T W_{2,T} L_T \left(L_T' W_{2,T} L_T\right)^{-1},$$

where $L_T = T^{-1}\sum_{t=1}^T \mathbb{E}m_{t\beta}(\beta_*)$ and $m_{t\beta}(\beta)$ is the $p\times k$ matrix of partial derivatives of $m_t(\beta)$, $W_{2,T}$ is a nonrandom matrix such that $\widehat{W}_{2,T} - W_{2,T} \overset{\mathbb{P}}{\to} 0$, and $J_T = T^{-1}\sum_{s=1}^T\sum_{t=1}^T \mathbb{E}(m_t(\beta_*)\, m_s(\beta_*))'$. Let $J = \lim_{T\to\infty} J_T$. The consistent estimation of H boils down to the consistent estimation of J since the

*Department of Economics and Finance, University of Rome Tor Vergata, Via Columbia 2, Rome 00133, IT. Email: alessandro.casini@uniroma2.it.

estimation of L_T and $W_{2,T}$ is straightforward. $\widehat{W}_{2,T}$ is a natural estimator of $W_{2,T}$ while under regularity conditions $L_T - T^{-1}\sum_{t=1}^{T} m_{t\beta}(\widehat{\beta}) \xrightarrow{\mathbb{P}} 0$. In place of the classical HAC estimators we now estimate J by

$$\widehat{J}_T = \sum_{k=-T+1}^{T-1} K_1 (b_{1,T} k) \widehat{\Gamma}(k), \quad \text{where} \quad \widehat{\Gamma}(k) \triangleq \frac{n_T}{T-n_T} \sum_{r=0}^{\lfloor (T-n_T)/n_T \rfloor} \widehat{c}_T (rn_T/T,\, k), \tag{N.1}$$

where

$$\widehat{c}_T (rn_T/T,\, k) \triangleq \begin{cases} (Tb_{2,T})^{-1} \sum_{s=k+1}^{T} K_2^* \left(\frac{((r+1)n_T - (s+k/2))/T}{b_{2,T}} \right) \widehat{m}_s \widehat{m}'_{s-k}, & k \geq 0 \\ (Tb_{2,T})^{-1} \sum_{s=-k+1}^{T} K_2^* \left(\frac{((r+1)n_T - (s-k/2))/T}{b_{2,T}} \right) \widehat{m}_{s+k} \widehat{m}'_s, & k < 0 \end{cases},$$

and $\widehat{m}_s = m_s(\widehat{\beta})$. We can implement $\widehat{J}_T$ with the data-dependent methods for selecting $b_{1,T}$ and $b_{2,T}$, and choose K_1 and K_2 on the basis of the optimality results of Section 4. For K_1 one can use the QS kernel while for K_2 one can choose $K_2 = 6x(1-x)$ for $0 \leq x \leq 1$ and 0 otherwise as suggested in Section 4. From the results in Section 5,

$$\widehat{b}_{1,T} = 0.6828 \left(\widehat{\phi}(2)\, T \widehat{\bar{b}}_{2,T} \right)^{-1/5}$$
$$\widehat{b}_{2,T}(u_r) = 1.6786 \left(\widehat{D}_1 (u_r) \right)^{-1/5} \left(\widehat{D}_2 (u_r) \right)^{1/5} T^{-1/5}, \qquad u_r = rn_T/T,$$

where the expressions for $\widehat{\phi}(2)$, $\widehat{D}_1(u_r)$ and $\widehat{D}_2(u_r)$ are given in the same section.

N.A.2 IV

Consider the linear model $y_t = x_t' \beta_0 + e_t$ $(t = 1, \ldots, T)$, where $\beta_0 \in \Theta \subset \mathbb{R}^p$, y_t is an observation on the dependent variable, x_t is a p-vector of regressors and e_t is an unobserved disturbance potentially autocorrelated. Suppose the regressor is endogenous: $\mathbb{E}(x_t e_t) \neq 0$. The IV estimator $\widehat{\beta}_{\text{IV}}$ is given by $\widehat{\beta}_{\text{IV}} = (Z'X)^{-1} Z'Y$, where $Y = (y_1, \ldots, y_T)'$, $X = (x_1, \ldots, x_T)'$ and $Z = (z_1, \ldots, z_T)'$ where z_t is a p-vector of instruments. The asymptotic variance of the IV estimator is given by the limit of $\text{Var}(\sqrt{T}(\widehat{\beta}_{\text{IV}} - \beta_0)) = Q_{ZX}^{-1} J_T Q_{ZX}^{-1}$ where $Q_{ZX} = T^{-1}\sum_{t=1}^{T} z_t x_t'$ and $J_T = T^{-1} \sum_{s=1}^{T} \sum_{t=1}^{T} \mathbb{E}(e_s z_s (e_t z_t)')$. A natural estimator of $\lim_{T\to\infty} Q_{ZX}$ is $T^{-1}\sum_{t=1}^{T} z_t x_t'$. Let $J = \lim_{T\to\infty} J_T$. J can be consistently estimated by $\widehat{J}_T$ as given in (N.1) where $\widehat{m}_t$ is replaced by $\widehat{e}_t z_t$ where $\widehat{e}_t = y_t - x_t' \widehat{\beta}_{\text{IV}}$.

N.B Appendix: Proofs of the Results of Section 2 and 4-5

N.B.1 Proofs of the Results of Section 2.1

N.B.1.1 Proof of Theorem 2.1

For $Tu \notin \mathcal{T}$ we use the arguments in the proof of Theorem 2.2 in Dahlhaus (1997). Without loss of generality, assume $T_{j-1}^0 < Tu < T_j^0$ for some $1 \leq j \leq m_0 + 1$. Then,

$$f_{j,T}(u,\, \omega) = \frac{1}{2\pi} \sum_{s=-\infty}^{\infty} \exp(-i\omega s) \int_{-\pi}^{\pi} \exp(i\eta s)\, A_{j,\lfloor Tu - s/2 \rfloor, T}^0 (\eta)\, \overline{A_{j,\lfloor Tu + s/2 \rfloor, T}^0 (\eta)} d\eta,$$

and

$$f_j(u,\,\omega)=\frac{1}{2\pi}\sum_{s=-\infty}^{\infty}\exp(-i\omega s)\int_{-\pi}^{\pi}\exp(i\eta s)\,A_j(u,\,\eta)\,\overline{A_j(u,\,\eta)}d\mu.$$

We have, in virtue of standard orthogonality relations,

$$\begin{aligned}&\int_{-\pi}^{\pi}\left|f_{j,T}(u,\,\omega)-f_j(u,\,\omega)\right|^2 d\omega\\&=\int_{-\pi}^{\pi}\Bigg|\frac{1}{2\pi}\sum_{s=-\infty}^{\infty}\exp(-i\omega s)\\&\times\left[\int_{-\pi}^{\pi}\exp(i\eta s)\left(A^0_{j,\lfloor Tu-s/2\rfloor,T}(\eta)\,\overline{A^0_{j,\lfloor Tu+s/2\rfloor,T}(\eta)}-A_j(u,\,\eta)\,\overline{A_j(u,\,\eta)}\right)d\eta\right]\Bigg|^2 d\omega\\&=\frac{1}{2\pi}\sum_{s=-\infty}^{\infty}|c_{s,j}|^2+o(1),\end{aligned}$$

where $c_{s,j}=\int_{-\pi}^{\pi}\exp(i\eta s)\,G_j(s/2T,\,\eta)\,d\eta$ and

$$G_j\left(\frac{s}{2T},\,\eta\right)=A_j\left(u-\frac{s}{2T},\,\eta\right)A_j\left(u+\frac{s}{2T},\,-\eta\right)-A_j(u,\,\eta)\,A_j(u,\,-\eta).$$

By well-known results on Fourier coefficients [cf. Bary (1964), Chapter 2.3], $|c_{s,j}|\leq Cs^{-\vartheta}$ and thus $\sum_{s=n}^{\infty}|c_{s,j}|^2=O(n^{1-2\vartheta})$. Let $\Delta_s(\omega)=\sum_{r=0}^{s-1}\exp(-i\omega r)$. Applying summation by parts yields

$$\begin{aligned}\sum_{s=0}^{n-1}|c_{s,j}|^2&=\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\sum_{s=0}^{n-1}\exp(-i(\omega-\eta)s)\,G_j\left(\frac{s}{2T},\,\omega\right)\overline{G_j\left(\frac{s}{2T},\,\eta\right)}d\omega d\eta\\&\leq\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\Bigg|-\sum_{s=0}^{n-1}[G_j\left(\frac{s}{2T},\,\omega\right)\overline{G_j\left(\frac{s}{2T},\,\eta\right)}-G_j\left(\frac{s-1}{2T},\,\omega\right)\overline{G_j\left(\frac{s-1}{2T},\,\eta\right)}]\Delta_s(\eta-\omega)\\&\quad+G_j\left(\frac{n-1}{2T},\,\omega\right)\overline{G_j\left(\frac{n-1}{2T},\,\eta\right)}\Delta_n(\eta-\omega)\Bigg|d\omega d\eta\\&=O\left(\frac{n\ln n}{T^{\vartheta}}\right).\end{aligned}$$

A similar bound holds for $\sum_{s=n}^{\infty}|c_{-s,j}|^2$. The result for $Tu\notin\mathcal{T}$ follows by choosing n appropriately. Next, suppose $Tu\in\mathcal{T}$ and $u=T_j^0/T$. Then, we have

$$f_{j,T}(u,\,\omega)=\frac{1}{2\pi}\sum_{s=-\infty}^{\infty}\exp(-i\omega s)\int_{-\pi}^{\pi}\exp(i\eta s)\,A^0_{j,\lfloor Tu-3|s|/2\rfloor,T}(\eta)\,\overline{A^0_{j,\lfloor Tu-|s|/2\rfloor,T}(\eta)}d\eta$$

and

$$f_j(u,\,\omega)=\frac{1}{2\pi}\sum_{s=-\infty}^{\infty}\exp(-i\omega s)\int_{-\pi}^{\pi}\exp(i\eta s)\,A_j(u,\,\eta)\,\overline{A_j(u,\,\eta)}d\eta.$$

Proceeding as above,

$$\int_{-\pi}^{\pi}\left|f_T(u,\,\omega)-f(u,\,\omega)\right|^2 d\omega$$

$$= \int_{-\pi}^{\pi} \left| \frac{1}{2\pi} \sum_{s=-\infty}^{\infty} \exp(-i\omega s) \right.$$

$$\left. \left[\int_{-\pi}^{\pi} \exp(i\eta s) A^0_{j,\lfloor uT-3|s|/2 \rfloor,T}(\eta) \overline{A^0_{j,\lfloor uT-|s|/2 \rfloor,T}(\eta)} d\eta - \int_{-\pi}^{\pi} \exp(i\eta s) A_j(u,\eta) \overline{A_j(u,\eta)} d\eta \right] \right|^2 d\omega$$

$$= \int_{-\pi}^{\pi} \left| \frac{1}{2\pi} \sum_{s=-\infty}^{\infty} \exp(-i\omega s) \right.$$

$$\left. \left[\int_{-\pi}^{\pi} \exp(i\eta s) \left(A^0_{j,\lfloor Tu-3|s|/2 \rfloor,T}(\eta) \overline{A^0_{j,\lfloor Tu-|s|/2 \rfloor,T}(\eta)} - A_j(u,\eta) \overline{A_j(u,\eta)} \right) d\eta \right] \right|^2 d\omega$$

$$= \frac{1}{2\pi} \sum_{s=-\infty}^{\infty} |c_{s,j}|^2 + o(1),$$

with $c_{s,j} = \int_{-\pi}^{\pi} \exp(i\eta s) G_j(s/2T, \eta) d\eta$ and

$$G_j\left(\frac{s}{2T}, \eta\right) = A_j\left(u - \frac{3|s|}{2T}, \eta\right) A_j\left(u - \frac{|s|}{2T}, -\eta\right) - A_j(u, \eta) A_j(u, -\eta).$$

Using the definition of $\Delta_s(\omega)$ and the above-mentioned properties of $c_{s,j}$ which continue to hold, summation by parts and the Lipschitz continuity of $A_j(u, \cdot)$ then imply $\sum_{s=0}^{n-1} |c_{s,j}|^2 = O(n \ln n / T^{\vartheta})$. Since the same bound applies to $\sum_{s=n}^{\infty} |c_{-s,j}|^2$, we can choose an appropriate n to yield the result for $Tu \in \mathcal{T}$. □

N.B.2 Proofs of the Results of Section 4

N.B.2.1 Proof of Proposition 4.1

We first need to show that $\sqrt{Tb_{2,T}} (\widehat{c}_T(rn_T/T, k) - \widetilde{c}(rn_T/T, k)) = o_{\mathbb{P}}(1)$. Without loss of generality, we can focus on the scalar case. From (S.27), $\left\| \frac{\partial}{\partial \beta'} \widehat{c}_T(rn_T/T, k) \right\| |_{\beta=\bar{\beta}} = O_{\mathbb{P}}(1)$. A mean-value Taylor's expansion gives

$$\begin{aligned}
\sqrt{Tb_{2,T}} (\widehat{c}_T(rn_T/T, k) - \widetilde{c}_T(rn_T/T, k)) &= \sqrt{b_{2,T}} \frac{\partial}{\partial \beta'} \widehat{c}_T(rn_T/T, k) |_{\beta=\bar{\beta}} \sqrt{T} \left(\widehat{\beta} - \beta_0 \right) \\
&\leq \sqrt{b_{2,T}} \sup_{r \geq 1} \left\| \frac{\partial}{\partial \beta'} \widehat{c}(rn_T/T, k) \right\| |_{\beta=\bar{\beta}} \sqrt{T} \left(\widehat{\beta} - \beta_0 \right) \\
&= \sqrt{b_{2,T}} O_{\mathbb{P}}(1) = o_{\mathbb{P}}(1).
\end{aligned}$$

Thus,

$$\xi_T = \text{vec}(\widehat{c}_T(rn_T/T, k) - \widetilde{c}(rn_T/T, k))' \widetilde{W}_T \text{vec}(\widehat{c}_T(rn_T/T, k) - \widetilde{c}(rn_T/T, k)) \xrightarrow{\mathbb{P}} 0.$$

Since ξ_T is a bounded sequence, $\mathbb{E}(\xi_T) \xrightarrow{\mathbb{P}} 0$. Hence, given that $\widetilde{W}_T \xrightarrow{\mathbb{P}} \widetilde{W}$, we have MSE$(1, \widehat{c}_T(u_0, k), \widetilde{W}_T) =$ MSE$(1, \widetilde{c}_T(u_0, k), \widetilde{W}) + o_{\mathbb{P}}(1)$. By using the results of Lemma S.A.4, the MSE of $\widehat{c}_T(u_0, k)$ for any $u_0 \in (0, 1)$ and any integer k, is given by

$$\begin{aligned}
&\mathbb{E}[\widehat{c}_T(u_0, k) - c(u_0, k)]^2 \\
&\quad = \frac{1}{4} b_{2,T}^4 \left(\int_0^1 x^2 K_2(x) dx \right)^2 \left(\frac{\partial^2}{\partial^2 u} c(u_0, k) \right)^2
\end{aligned}$$

$$+\frac{1}{Tb_{2,T}}\int_0^1 K_2^2(x)\,dx\sum_{l=-\infty}^{\infty}c(u_0,\,l)\left[c(u_0,\,l)+c(u_0,\,l+2k)\right]$$

$$+\frac{1}{Tb_{2,T}}\int_0^1 K_2^2(x)\,dx\sum_{h_1=-\infty}^{\infty}\kappa_{V,\lfloor Tu_0\rfloor}(-k,\,h_1,\,h_1-k)+o\left(b_{2,T}^4\right)+o\left(1/\left(b_{2,T}T\right)\right)$$

$$\triangleq g(K_2,\,b_{2,T})+o\left(b_{2,T}^4\right)+o\left(1/\left(b_{2,T}T\right)\right). \tag{N.1}$$

Then $g(K_2,\,b_{2,T})=4^{-1}b_{2,T}^4H(K_2)\,D_1(u_0)+(Tb_{2,T})^{-1}F(K_2)\,(D_2(u_0)+D_3(u_0))$. The minimum of $g(K_2,\,b_{2,T})$ in $b_{2,T}$ is determined by the equation

$$\frac{\partial}{\partial b_{2,T}}g(K_2,\,b_{2,T})=b_{2,T}^3H(K_2)\,D_1(u_0)-\frac{1}{Tb_{2,T}^2}F(K_2)\,(D_2(u_0)+D_3(u_0))=0.$$

The minimum is achieved at

$$b_{2,T}^{\mathrm{opt}}=\left[H(K_2)\,D_1(u_0)\right]^{-1/5}\left(F(K_2)\,(D_2(u_0)+D_3(u_0))\right)^{1/5}T^{-1/5}.$$

If $V_{t,T}$ is Gaussian, then the term involving $\kappa_{V,\lfloor Tu_0\rfloor}$ in (N.1) is equal to zero and so $D_3(u_0)=0$ in $b_{2,T}^{\mathrm{opt}}$. Next, we minimize $g(K_2,\,b_{2,T}^{\mathrm{opt}})$ with respect to the class of kernels $K_2:\mathbb{R}\to[0,\,\infty]$ that are centered at $x=1/2$ with

$$\int_{\mathbb{R}}K_2(x)\,dx=1, \tag{N.2}$$

$$K_2(x)=K_2(1-x). \tag{N.3}$$

We use arguments similar to those in Chapter 7 of Priestley (1981) and in Dahlhaus and Giraitis (1998). Let

$$\sqrt{K_{2\sigma}(x)}=\frac{1}{\sqrt{\sigma}}\left(K_2\left(\frac{x-1/2}{\sigma}+\frac{1}{2}\right)\right)^{1/2},\qquad \text{where } \sigma\in(0,\,\infty).$$

We have $F(K_{2\sigma})=(1/\sigma)\,F(K_2)$ and $H(K_{2\sigma})=\sigma^4H(K_2)$ (with the integrals in the definition of F and H extended to $\mathbb{R}$ and with the variable of integration x subtracted by $1/2$). Then, $b_{2,K_{2\sigma},T}^{\mathrm{opt}}=\sigma^{-1}b_{2,T}^{\mathrm{opt}}$ where $b_{2,K_{2\sigma},T}^{\mathrm{opt}}$ is the optimal bandwidth associated with the kernel $K_{2\sigma}$. Also, $g(K_{2\sigma},\,b_{2,K_{2\sigma},T}^{\mathrm{opt}})=g(K_2,\,b_{2,T}^{\mathrm{opt}})$. We can thus restrict our attention to K_2 satisfying

$$\int_{\mathbb{R}}\left(x-\frac{1}{2}\right)^2K_2(x)\,dx=\int_{\mathbb{R}}\left(x-\frac{1}{2}\right)^2K_2^{\mathrm{opt}}(x)\,dx, \tag{N.4}$$

where $K_2^{\mathrm{opt}}(x)=6x(1-x)$ for $x\in[0,\,1]$ and $K_2^{\mathrm{opt}}(x)=0$ for $x\notin[0,\,1]$. Therefore, we have to show that, for any K_2 that satisfies (N.2)-(N.3),

$$\int_{\mathbb{R}/[0,\,1]}K_2^2(x)\,dx+\int_0^1K_2^2(x)\,dx=\int_{\mathbb{R}}K_2^2(x)\,dx\geq\int_{\mathbb{R}}\left(K_2^{\mathrm{opt}}(x)\right)^2dx=\int_0^1\left(K_2^{\mathrm{opt}}(x)\right)^2dx.$$

This is implied by

$$\int_0^1K_2^2(x)\,dx\geq\int_0^1\left(K_2^{\mathrm{opt}}(x)\right)^2dx.$$

Let $K_2(x) = K_2^{\text{opt}}(x) + \varepsilon(x)$, $x \in \mathbb{R}$, where $\varepsilon > 0$. Since $\int_{\mathbb{R}} \varepsilon^2(x)\,dx \geq 0$ and K_2^{opt} vanishes outside $[0,\,1]$, it is sufficient to prove that $\int_0^1 \left(K_2^{\text{opt}}(x)\,\varepsilon(x)\right) dx \geq 0$ because

$$\int_0^1 K_2^2(x)\,dx = \int_0^1 \left(K_2^{\text{opt}}(x) + \varepsilon(x)\right)^2 dx \geq \int_0^1 \left(K_2^{\text{opt}}(x)\right)^2 + 2\int_0^1 \left(K_2^{\text{opt}}(x)\,\varepsilon(x)\right) dx.$$

By (N.2), we have $\int_{\mathbb{R}} \varepsilon(x)\,dx = 0$, while $\int_{\mathbb{R}} \varepsilon(x)\,(x^2 - x)\,dx = 0$ in view of

$$\begin{aligned}0 = \int_{\mathbb{R}} \left(K_2(x) - K_2^{\text{opt}}(x)\right)\left(x - \frac{1}{2}\right)^2 dx &= \int_{\mathbb{R}} \left(K_2(x) - K_2^{\text{opt}}(x)\right)\left(x^2 - x\right) dx + \frac{1}{4}\int_{\mathbb{R}} \varepsilon(x)\,dx \\ &= \int_{\mathbb{R}} \left(K_2(x) - K_2^{\text{opt}}(x)\right)\left(x^2 - x\right) dx.\end{aligned}$$

Note that $(x^2 - x) = x(x-1)$. Therefore, we deduce

$$6\int_{\mathbb{R}/[0,\,1]} x(1-x)\,\varepsilon(x)\,dx + 6\int_0^1 x(1-x)\,\varepsilon(x)\,dx = 0.$$

Rearranging the last expression yields,

$$\int_0^1 K_2^{\text{opt}}(x)\,\varepsilon(x)\,dx = 6\int_{\mathbb{R}/[0,\,1]} x(x-1)\,\varepsilon(x)\,dx \geq 0,$$

because $\varepsilon(x) \geq 0$ and $x(x-1) \geq 0$ for $x \notin [0,\,1]$. $\square$

N.B.2.2 Proof of Theorem 4.1

Without loss of generality, we provide the proof for the scalar case. By Theorem 3.2-(iii), if $Tb_{1,T}^{2q+1} b_{2,T} \to \gamma_2 \in (0,\,\infty)$ for some $q \in [0,\,\infty)$ for which $K_{1,q}$, $|\int_0^1 f^{(q)}(u,\,0)\,du| \in [0,\,\infty)$, then

$$\begin{aligned}&\lim_{T\to\infty} \text{MSE}\left(Tb_{1,T}b_{2,T},\,\widehat{J}_T(b_{1,T,K_1}),\,1\right) \\ &\quad = 4\pi^2\left[\gamma_2 K_{1,q}^2 \left(\int_0^1 f^{(q)}(u,\,0)\,du\right)^2 + \int K_1^2(y)\,dy \int_0^1 (K_2(x))^2\,dx \left(\int_0^1 f(u,\,0)\,du\right)^2\right].\end{aligned}$$

We have $Tb_{1,T}^5 b_{2,T} \to \gamma$ by assumption. Thus, we apply Theorem 3.2-(iii) with $q = 2$, K_1 and b_{1,T,K_1}. Then, $Tb_{1,T,K_1}^5 b_{2,T} \to \gamma/\left(\int K_1^2(x)\,dx\right)^5$ and

$$Tb_{1,T}b_{2,T} = Tb_{1,T,K_1}b_{2,T}\int K_1^2(x)\,dx.$$

Therefore, given $K_{1,2} < \infty$,

$$\begin{aligned}&\lim_{T\to\infty}\left(\text{MSE}\left(Tb_{1,T}b_{2,T},\,\widehat{J}_T(b_{1,T,K_1}),\,1\right) - \text{MSE}\left(Tb_{1,T}b_{2,T},\,\widehat{J}_T^{\text{QS}}(b_{1,T}),\,1\right)\right) \\ &\quad = 4\gamma\pi^2\left(\int_0^1 f^{(q)}(u,\,0)\,du\right)^2 \int_0^1 (K_2(x))^2\,dx\left[K_{1,2}^2\left(\int K_1^2(y)\,dy\right)^4 - \left(K_{1,2}^{\text{QS}}\right)^2\right].\end{aligned}$$

Let $\widetilde{K}_1(\cdot)$ and $\widetilde{K}_1^{\mathrm{QS}}(\cdot)$ denote the spectral window generators of $K_1(\cdot)$ and $K_1^{\mathrm{QS}}(\cdot)$, respectively. They have the following properties: $K_{1,2}=\int_{-\infty}^{\infty}\omega^2\widetilde{K}_1(\omega)\,d\omega$, $K_1(0)=\int_{-\infty}^{\infty}\widetilde{K}_1(\omega)\,d\omega$, and $\int_{-\infty}^{\infty}K_1^2(x)\,dx=\int_{-\infty}^{\infty}\widetilde{K}_1^2(\omega)\,d\omega$. As in Andrews (1991), the result of the theorem follows if we can show the following inequality,

$$K_{1,2}^2\left(\int K_1^2(x)\,dx\right)^4\geq\left(K_{1,2}^{\mathrm{QS}}\right)^2\qquad\text{for all } K_1(\cdot)\in\boldsymbol{K}_1. \tag{N.5}$$

Priestley (1981, Ch. 7.5) showed that $\widetilde{K}_1^{\mathrm{QS}}(\cdot)$ minimizes

$$\int_{-\infty}^{\infty}\omega^2\widetilde{K}_1(\omega)\,d\omega\left(\int_{-\infty}^{\infty}\widetilde{K}_1^2(\omega)\,d\omega\right)^2, \tag{N.6}$$

subject to (a) $\int_{-\infty}^{\infty}\widetilde{K}_1(\omega)\,d\omega=1$, (b) $\widetilde{K}_1(\omega)\geq 0$, $\forall\,\omega\in\mathbb{R}$, and (c) $\widetilde{K}_1(\omega)=\widetilde{K}_1(-\omega)$, $\forall\,\omega\in\mathbb{R}$, where $K_1^{\mathrm{QS}}(\omega)=(5/8\pi)\left(1-\omega^2/c^2\right)$ for $|\omega|\leq c$ for $c=6\pi/5$. and $K_1^{\mathrm{QS}}(\omega)=0$ otherwise. Note that the inequality (N.5) holds if and only if $\widetilde{K}_1^{\mathrm{QS}}(\cdot)$ minimizes (N.6). This proves the inequality of the theorem. Strict inequality holds when $K_1^{\mathrm{QS}}(x)\neq K_1(x)$ with positive Lebesgue measure. □

N.B.2.3 Proof of Corollary 4.1

Note that $T^{\frac{2q}{2q+1}}b_{2,T}^{\frac{2q}{2q+1}}=(Tb_{1,T}^{2q+1}b_{2,T})^{-1/(2q+1)}Tb_{1,T}b_{2,T}=(\gamma^{-1/(2q+1)}+o(1))Tb_{1,T}b_{2,T}$. Thus,

$$\begin{aligned}&\lim_{T\to\infty}\mathrm{MSE}\left(T^{\frac{2q}{2q+1}}b_{2,T}^{\frac{2q}{2q+1}},\,\widehat{J}_T(b_{1,T},\,b_{2,T}),\,W_T\right)\\&\quad=\gamma^{-1/(2q+1)}4\pi^2\left[\gamma K_{1,q}^2\mathrm{vec}\left(\int_0^1 f^{(q)}(u,\,0)\,du\right)'W\mathrm{vec}\left(\int_0^1 f^{(q)}(u,\,0)\,du\right)\right.\\&\qquad\left.+\int K_1^2(y)\,dy\int_0^1 K_2^2(x)\,dx\,\mathrm{tr}W\left(I_{p^2}-C_{pp}\right)\left(\int_0^1 f(u,\,0)\,du\right)\otimes\left(\int_0^1 f(v,\,0)\,dv\right)\right].\end{aligned} \tag{N.7}$$

Minimizing this with respect to γ gives

$$\begin{aligned}&\gamma^{2q/(2q+1)}K_{1,q}^2\mathrm{vec}\left(\int_0^1 f^{(q)}(u,\,0)\,du\right)'W\mathrm{vec}\left(\int_0^1 f^{(q)}(u,\,0)\,du\right)\\&\quad=\gamma^{-1/(2q+1)}\int K_1^2(y)\,dy\int K_2^2(x)\,dx\,\mathrm{tr}W\left(I_{p^2}-C_{pp}\right)\left(\int_0^1 f(u,\,0)\,du\right)\otimes\left(\int_0^1 f(v,\,0)\,dv\right),\end{aligned}$$

or

$$\begin{aligned}\gamma^{\mathrm{opt}}&=\frac{1}{2q}\frac{\int K_1^2(y)\,dy\int K_2^2(x)\,dx\,\mathrm{tr}W\left(I_{p^2}+C_{pp}\right)\left(\int_0^1 f(u,\,0)\,du\right)\otimes\left(\int_0^1 f(v,\,0)\,dv\right)}{K_{1,q}^2\mathrm{vec}\left(\int_0^1 f^{(q)}(u,\,0)\,du\right)'W\mathrm{vec}\left(\int_0^1 f^{(q)}(u,\,0)\,du\right)}\\&=\left(2qK_{1,q}^2\phi(q)\right)^{-1}\left(\int K_1^2(y)\,dy\int_0^1 K_2^2(x)\,dx\right).\end{aligned}$$

Note that $\gamma^{\mathrm{opt}}>0$ provided that $0<\phi(q)<\infty$ and W is positive definite. Hence, $\{b_{1,T}\}$ is optimal in the sense that $Tb_{1,T}^{2q+1}b_{2,T}\to\gamma^{\mathrm{opt}}$ if and only if $b_{1,T}=b_{1,T}^{\mathrm{opt}}+o((Tb_{2,T})^{-1/(2q+1)})$ where $b_{2,T}=O(b_{2,T}^{\mathrm{opt}})$. □

N.B.3 Proofs of the Results of Section 5

N.B.3.1 Proof of Theorem 5.1

Without loss of generality, we assume that V_t is a scalar. The constant $C < \infty$ may vary from line to line. We begin with the proof of part (ii) because it becomes then simpler to prove part (i). By Theorem 3.2-(ii), $\sqrt{Tb_{\theta_1,T}b_{\theta_2,T}}(\widehat{J}_T(b_{\theta_1,T},\, b_{\theta_2,T}) - J_T) = O_{\mathbb{P}}(1)$. It remains to establish the second result of Theorem 5.1-(ii). Let $S_T = \left\lfloor b_{\theta_1,T}^{-r} \right\rfloor$ where

$$\begin{aligned} r \in &(\max\{(8b-5-2q)/8(b-1),\, 1.25,\, (b/2-1/4)/(b-1),\, q/(l-1),\, (8b-7-6q)/8(b-1) \\ &(b-3/4-q/2)/(b-1)\},\, \min\{13q/24+49/48,\, 46/48+20q/48,\, 7/8+3q/4,\, (6+4q)/8,\, 2\}), \end{aligned}$$

with $b > 1 + 1/q$. We will use the following decomposition

$$\begin{aligned} \widehat{J}_T(\widehat{b}_{1,T},\, \widehat{\overline{b}}_{2,T}) - \widehat{J}_T(b_{\theta_1,T},\, b_{\theta_2,T}) &= (\widehat{J}_T(\widehat{b}_{1,T},\, \widehat{\overline{b}}_{2,T}) - \widehat{J}_T(b_{\theta_1,T},\, \widehat{\overline{b}}_{2,T})) \\ &\quad + (\widehat{J}_T(b_{\theta_1,T},\, \widehat{\overline{b}}_{2,T}) - \widehat{J}_T(b_{\theta_1,T},\, b_{\theta_2,T})). \end{aligned} \tag{N.8}$$

Let

$$\begin{aligned} N_1 &\triangleq \{-S_T,\, -S_T+1, \ldots,\, -1,\, 1, \ldots,\, S_T-1,\, S_T\}, \\ N_2 &\triangleq \{-T+1, \ldots,\, -S_T-1,\, S_T+1, \ldots,\, T-1\}. \end{aligned}$$

Let us consider the first term of (N.8). We have

$$\begin{aligned} T^{8q/10(2q+1)}&(\widehat{J}_T(\widehat{b}_{1,T},\, \widehat{\overline{b}}_{2,T}) - \widehat{J}_T(b_{\theta_1,T},\, \widehat{\overline{b}}_{2,T})) \\ &= T^{8q/10(2q+1)} \sum_{k\in N_1} (K_1(\widehat{b}_{1,T}k) - K_1(b_{\theta_1,T}k))\widehat{\Gamma}(k) \\ &\quad + T^{8q/10(2q+1)} \sum_{k\in N_2} K_1(\widehat{b}_{1,T}k)\widehat{\Gamma}(k) \\ &\quad - T^{8q/10(2q+1)} \sum_{k\in N_2} K_1(b_{\theta_1,T}k)\widehat{\Gamma}(k) \\ &\triangleq A_{1,T} + A_{2,T} - A_{3,T}. \end{aligned} \tag{N.9}$$

We first show that $A_{1,T} \xrightarrow{\mathbb{P}} 0$. Let $A_{1,1,T}$ denote $A_{1,T}$ with the summation restricted over positive integers k. Let $\widetilde{n}_T = \inf\{T/n_{3,T},\, \sqrt{n_{2,T}}\}$. We can use the Liptchitz condition on $K_1(\cdot) \in \boldsymbol{K}_3$ to yield,

$$\begin{aligned} |A_{1,1,T}| &\leq T^{8q/10(2q+1)} \sum_{k=1}^{S_T} C_2 \left|\widehat{b}_{1,T} - b_{\theta_1,T}\right| k \left|\widehat{\Gamma}(k)\right| \\ &\leq C\widetilde{n}_T \left|\widehat{\phi}(q)^{1/(2q+1)} - \phi_{\theta^*}^{1/(2q+1)}\right| \left(\widehat{\phi}(q)\,\phi_{\theta^*}\right)^{-1/(2q+1)} \widehat{\overline{b}}_{2,T}^{-1/(2q+1)} T^{(8q-10)/10(2q+1)} \widetilde{n}_T^{-1} \sum_{k=1}^{S_T} k \left|\widehat{\Gamma}(k)\right|, \end{aligned} \tag{N.10}$$

for some $C < \infty$. By Assumption 5.1-(ii) ($\widetilde{n}_T \left|\widehat{\phi}(q) - \phi_{\theta^*}\right| = O_{\mathbb{P}}(1)$) and, using the delta method, it suffices to show that $B_{1,T} + B_{2,T} + B_{3,T} \xrightarrow{\mathbb{P}} 0$, where

$$B_{1,T} = \widehat{\overline{b}}_{2,T}^{-1/(2q+1)} T^{(8q-10)/10(2q+1)} \widetilde{n}_T^{-1} \sum_{k=1}^{S_T} k \left|\widehat{\Gamma}(k) - \widetilde{\Gamma}(k)\right|, \tag{N.11}$$

$$B_{2,T}=\widehat{\overline{b}}_{2,T}^{-1/(2q+1)}T^{(8q-10)/10(2q+1)}\widetilde{n}_T^{-1}\sum_{k=1}^{S_T}k\left|\widetilde{\Gamma}(k)-\Gamma_T(k)\right|,$$

$$B_{3,T}=\widehat{\overline{b}}_{2,T}^{-1/(2q+1)}T^{(8q-10)/10(2q+1)}\widetilde{n}_T^{-1}\sum_{k=1}^{S_T}k\left|\Gamma_T(k)\right|,$$

with $\Gamma_T(k)\triangleq(n_T/T)\sum_{r=0}^{\lfloor T/n_T\rfloor}c(rn_T/T,\,k)$. By a mean-value expansion, we have

$$\begin{aligned}B_{1,T}&\le\widehat{\overline{b}}_{2,T}^{-1/(2q+1)}T^{(8q-10)/10(2q+1)}\widetilde{n}_T^{-1}T^{-1/2}\sum_{k=1}^{S_T}k\left|\left(\frac{\partial}{\partial\beta'}\widehat{\Gamma}(k)\,|_{\beta=\overline{\beta}}\right)\sqrt{T}\left(\widehat{\beta}-\beta_0\right)\right|\\&\le C\widehat{\overline{b}}_{2,T}^{-1/(2q+1)}T^{(8q-10)/10(2q+1)-1/2}\left(Tb_{\theta_2,T}\right)^{2r/(2q+1)}\widetilde{n}_T^{-1}\sup_{k\ge1}\left\|\frac{\partial}{\partial\beta}\widehat{\Gamma}(k)\,|_{\beta=\overline{\beta}}\right\|\sqrt{T}\left\|\widehat{\beta}-\beta_0\right\|\\&\le C\widehat{\overline{b}}_{2,T}^{(-1+2r)/(2q+1)}T^{(8q-10)/10(2q+1)-1/2+2r/(2q+1)}\widetilde{n}_T^{-1}\sup_{k\ge1}\left\|\frac{\partial}{\partial\beta}\widehat{\Gamma}(k)\,|_{\beta=\overline{\beta}}\right\|\sqrt{T}\left\|\widehat{\beta}-\beta_0\right\|\overset{\mathbb{P}}{\to}0,\end{aligned}\tag{N.12}$$

since $\widetilde{n}_T/T^{1/3}\to\infty$, $r<13q/24+49/48$, $\sqrt{T}||\widehat{\beta}-\beta_0||=O_{\mathbb{P}}(1)$, and $\sup_{k\ge1}||(\partial/\partial\beta)\widehat{\Gamma}(k)\,|_{\beta=\overline{\beta}}||=O_{\mathbb{P}}(1)$ using (S.27) and Assumption 3.3-(ii,iii). In addition,

$$\begin{aligned}\mathbb{E}\left(B_{2,T}^2\right)&\le\mathbb{E}\left(\widehat{\overline{b}}_{2,T}^{-2/(2q+1)}T^{(8q-10)/5(2q+1)}\widetilde{n}_T^{-2}\sum_{k=1}^{S_T}\sum_{j=1}^{S_T}kj\left|\widetilde{\Gamma}(k)-\Gamma_T(k)\right|\left|\widetilde{\Gamma}(j)-\Gamma_T(j)\right|\right)\\&\le\widehat{\overline{b}}_{2,T}^{-2/(2q+1)-1}T^{(8q-10)/5(2q+1)-2/3-1}S_T^4\sup_{k\ge1}T\widehat{\overline{b}}_{2,T}\mathrm{Var}\left(\widetilde{\Gamma}(k)\right)\\&\le\widehat{\overline{b}}_{2,T}^{-2/(2q+1)-1}T^{(8q-10)/5(2q+1)-2/3-1}\left(Tb_{\theta_2,T}\right)^{4r/(2q+1)}\sup_{k\ge1}T\widehat{\overline{b}}_{2,T}\mathrm{Var}\left(\widetilde{\Gamma}(k)\right)\\&\le T^{1/5}T^{2/5(2q+1)}T^{(8q-10)/5(2q+1)-2/3-1}T^{4r/(2q+1)}T^{-4r/5(2q+1)}\sup_{k\ge1}T\widehat{\overline{b}}_{2,T}\mathrm{Var}\left(\widetilde{\Gamma}(k)\right)\to0,\end{aligned}\tag{N.13}$$

given that $\sup_{k\ge1}T\widehat{\overline{b}}_{2,T}\mathrm{Var}(\widetilde{\Gamma}(k))=O(1)$ using Lemma S.A.5 and $r<46/48+20q/48$. Assumption 5.1-(iii) and $\sum_{k=1}^{\infty}k^{1-l}<\infty$ for $l>2$ yield

$$B_{3,T}\le\widehat{\overline{b}}_{2,T}^{-1/(2q+1)}T^{(8q-10)/10(2q+1)}\widetilde{n}_T^{-1}C_3\sum_{k=1}^{\infty}k^{1-l}\to0,\tag{N.14}$$

where we have used $\widetilde{n}_T/T^{3/10}\to\infty$ and $q<34/4$. Combining (N.10)-(N.14), we deduce that $A_{1,1,T}\overset{\mathbb{P}}{\to}0$. The same argument applied to $A_{1,T}$, where the summation now extends over negative integers k, gives $A_{1,T}\overset{\mathbb{P}}{\to}0$. Next, we show that $A_{2,T}\overset{\mathbb{P}}{\to}0$. Again, we use the notation $A_{2,1,T}$ (resp., $A_{2,2,T}$) to denote $A_{2,T}$ with the summation over positive (resp., negative) integers. Let $A_{2,1,T}=L_{1,T}+L_{2,T}+L_{3,T}$, where

$$\begin{aligned}L_{1,T}&=T^{8q/10(2q+1)}\sum_{k=S_T+1}^{T-1}K_1\left(\widehat{b}_{1,T}k\right)\left(\widehat{\Gamma}(k)-\widetilde{\Gamma}(k)\right),\\L_{2,T}&=L_{2,T}^A+L_{2,T}^B=T^{8q/10(2q+1)}\left(\sum_{k=S_T+1}^{\lfloor D_TT^{8/5(2q+1)}\rfloor}+\sum_{k=\lfloor D_TT^{8/5(2q+1)}\rfloor+1}^{T-1}\right)K_1\left(\widehat{b}_{1,T}k\right)\left(\widetilde{\Gamma}(k)-\Gamma_T(k)\right),\\L_{3,T}&=T^{8q/10(2q+1)}\sum_{k=S_T+1}^{T-1}K_1\left(\widehat{b}_{1,T}k\right)\Gamma_T(k).\end{aligned}\tag{N.15}$$

We apply a mean-value expansion and use $\sqrt{T}(\widehat{\beta}-\beta_0)=O_{\mathbb{P}}(1)$ as well as (S.27) to obtain

$$\begin{aligned}|L_{1,T}| &= T^{8q/10(2q+1)-1/2}\sum_{k=S_T+1}^{T-1} C_1\left(\widehat{b}_{1,T}k\right)^{-b}\left|\left(\frac{\partial}{\partial\beta'}\widehat{\Gamma}(k)\right)|_{\beta=\overline{\beta}}\sqrt{T}\left(\widehat{\beta}-\beta_0\right)\right| \\ &= T^{8q/10(2q+1)-1/2+4b/5(2q+1)}\sum_{k=S_T+1}^{T-1} C_1 k^{-b}\left|\left(\frac{\partial}{\partial\beta'}\widehat{\Gamma}(k)\right)|_{\beta=\overline{\beta}}\sqrt{T}\left(\widehat{\beta}-\beta_0\right)\right| \\ &= T^{8q/10(2q+1)-1/2+4b/5(2q+1)+4r(1-b)/5(2q+1)}\left|\left(\frac{\partial}{\partial\beta'}\widehat{\Gamma}(k)\right)|_{\beta=\overline{\beta}}\sqrt{T}\left(\widehat{\beta}-\beta_0\right)\right| \\ &= T^{8q/10(2q+1)-1/2+4b/5(2q+1)+4r(1-b)/5(2q+1)}O_{\mathbb{P}}(1)\,O_{\mathbb{P}}(1),\end{aligned} \tag{N.16}$$

which goes to zero since $r>(8b-5-2q)/8(b-1)$. Let us now consider $L_{2,T}$. We have

$$\begin{aligned}\left|L_{2,T}^{A}\right| &= T^{(8q-1)/10(2q+1)}\sum_{k=S_T+1}^{\lfloor D_T T^{8/5(2q+1)}\rfloor} C_1\left(\widehat{b}_{1,T}k\right)^{-b}\left|\widetilde{\Gamma}(k)-\Gamma_T(k)\right| \\ &= C_1\left(2qK_{1,q}^2\widehat{\phi}(q)\right)^{b/(2q+1)}T^{8q/10(2q+1)+b/(2q+1)-1/2}\widehat{\overline{b}}_{2,T}^{\,b/(2q+1)-1/2}\left(\sum_{k=S_T+1}^{\lfloor D_T T^{8/5(2q+1)}\rfloor} k^{-b}\right) \\ &\quad\times\sqrt{T\widehat{\overline{b}}_{2,T}}\left|\widetilde{\Gamma}(k)-\Gamma_T(k)\right|.\end{aligned} \tag{N.17}$$

Note that

$$\begin{aligned}&\mathbb{E}\left(T^{8q/10(2q+1)+b/(2q+1)-1/2}\widehat{\overline{b}}_{2,T}^{\,b/(2q+1)-1/2}\sum_{k=S_T+1}^{\lfloor D_T T^{8/5(2q+1)}\rfloor} k^{-b}\sqrt{T\widehat{\overline{b}}_{2,T}}\left|\widetilde{\Gamma}(k)-\Gamma_T(k)\right|\right)^2 \\ &\leq T^{8q/5(2q+1)+2b/(2q+1)-1}\widehat{\overline{b}}_{2,T}^{\,2b/(2q+1)-1}\left(\sum_{k=S_T+1}^{\lfloor D_T T^{8/5(2q+1)}\rfloor} k^{-b}\sqrt{T\widehat{\overline{b}}_{2,T}}\left(\mathrm{Var}\left(\widetilde{\Gamma}(k)\right)\right)^{1/2}\right)^2 \\ &= T^{8q/5(2q+1)+2b/(2q+1)-1}\widehat{\overline{b}}_{2,T}^{\,2b/(2q+1)-1}\left(\sum_{k=S_T+1}^{\lfloor D_T T^{8/5(2q+1)}\rfloor} k^{-b}\right)^2 O(1) \\ &= T^{8q/5(2q+1)+2b/(2q+1)-1}\widehat{\overline{b}}_{2,T}^{\,2b/(2q+1)-1} S_T^{2(1-b)}O(1)\to 0,\end{aligned} \tag{N.18}$$

since $r>1.25$ and $T\widehat{\overline{b}}_{2,T}\mathrm{Var}(\widetilde{\Gamma}(k))=O(1)$ as above. Next,

$$\begin{aligned}\left|L_{2,T}^{B}\right| &= T^{(8q-1)/10(2q+1)}\sum_{k=\lfloor D_T T^{8/5(2q+1)}\rfloor+1}^{T-1} C_1\left(\widehat{b}_{1,T}k\right)^{-b}\left|\widetilde{\Gamma}(k)-\Gamma_T(k)\right| \\ &= C_1\left(2qK_{1,q}^2\widehat{\phi}(q)\right)^{b/(2q+1)}T^{8q/10(2q+1)+b/(2q+1)-1/2}\widehat{\overline{b}}_{2,T}^{\,b/(2q+1)-1/2}\left(\sum_{k=\lfloor D_T T^{8/5(2q+1)}\rfloor+1}^{T-1} k^{-b}\right) \\ &\quad\times\sqrt{T\widehat{\overline{b}}_{2,T}}\left|\widetilde{\Gamma}(k)-\Gamma_T(k)\right|.\end{aligned} \tag{N.19}$$

Note that

$$\mathbb{E}\left(T^{8q/10(2q+1)+b/(2q+1)-1/2}\widehat{b}_{2,T}^{b/(2q+1)-1/2}\sum_{k=\left\lfloor D_T T^{8/5(2q+1)}\right\rfloor+1}^{T-1}k^{-b}\sqrt{T\widehat{\overline{b}}_{2,T}}\left|\widetilde{\Gamma}\left(k\right)-\Gamma_T\left(k\right)\right|\right)^2 \tag{N.20}$$

$$\leq T^{8q/5(2q+1)+2b/(2q+1)-1}\widehat{b}_{2,T}^{2b/(2q+1)-1}\left(\sum_{k=\left\lfloor D_T T^{8/5(2q+1)}\right\rfloor+1}^{T-1}k^{-b}\sqrt{T\widehat{\overline{b}}_{2,T}}\left(\mathrm{Var}\left(\widetilde{\Gamma}\left(k\right)\right)\right)^{1/2}\right)^2$$

$$= T^{8q/5(2q+1)+2b/(2q+1)-1}\widehat{b}_{2,T}^{2b/(2q+1)-1}\left(\sum_{k=\left\lfloor D_T T^{8/5(2q+1)}\right\rfloor+1}^{T-1}k^{-b}\right)^2 O\left(1\right)$$

$$= T^{8q/5(2q+1)+2b/(2q+1)-1}\widehat{b}_{2,T}^{2b/(2q+1)-1}S_T^{2(1-b)}T^{16(1-b)/5(2q+1)}D_T^2 O\left(1\right)\rightarrow 0,$$

since $r>\left(b/2-1/4\right)/\left(b-1\right)$. Combining (N.17) and (N.20) yields $L_{2,T}\xrightarrow{\mathbb{P}}0$, since $\widehat{\phi}\left(q\right)=O_{\mathbb{P}}\left(1\right)$. Let us turn to $L_{3,T}$. By Assumption 5.1-(iii) and $\left|K_1\left(\cdot\right)\right|\leq 1$, we have,

$$\left|L_{3,T}\right|\leq T^{8q/10(2q+1)}\sum_{k=S_T}^{T-1}C_3 k^{-l}\leq T^{8q/10(2q+1)}C_3 S_T^{1-l} \tag{N.21}$$

$$\leq C_3 T^{8q/10(2q+1)}T^{-4r(l-1)/5(2q+1)}\rightarrow 0,$$

since $r>q/\left(l-1\right)$. In view of (N.15)-(N.21), we deduce that $A_{2,1,T}\xrightarrow{\mathbb{P}}0$. Applying the same argument to $A_{2,2,T}$, we have $A_{2,T}\xrightarrow{\mathbb{P}}0$. Using similar arguments, one has $A_{3,T}\xrightarrow{\mathbb{P}}0$. It remains to show that $T^{8q/10(2q+1)}(\widehat{J}_T(b_{\theta_1,T},\,\widehat{\overline{b}}_{2,T})-\widehat{J}_T(b_{\theta_1,T},\,b_{\theta_2,T}))\xrightarrow{\mathbb{P}}0$. Let $\widehat{c}_{\theta_2,T}\left(rn_T/T,\,k\right)$ denote the estimator that uses $b_{\theta_2,T}$ in place of $\widehat{b}_{2,T}$. We have for $k\geq 0$,

$$\widehat{c}_T\left(rn_T/T,\,k\right)-\widehat{c}_{\theta_2,T}\left(rn_T/T,\,k\right)$$
$$=\left(T\overline{b}_{\theta_2,T}\right)^{-1}\sum_{s=k+1}^{T}\left(K_2\left(\frac{\left(\left(r+1\right)n_T-\left(s-k/2\right)\right)/T}{\widehat{b}_{2,T}\left(\left(r+1\right)n_T/T\right)}\right)-K_2\left(\frac{\left(\left(r+1\right)n_T-\left(s-k/2\right)\right)/T}{b_{\theta_2,T}\left(\left(r+1\right)n_T/T\right)}\right)\right)\widehat{V}_s\widehat{V}_{s-k}$$
$$+O_{\mathbb{P}}\left(1/T\overline{b}_{\theta_2,T}\right). \tag{N.22}$$

Given Assumption 5.1-(v,vi,vii) and using the delta method, we have for $s\in\{Tu-\lfloor Tb_{\theta_2,T}\rfloor,\ldots,\,Tu+\lfloor Tb_{\theta_2,T}\rfloor\}$:

$$K_2\left(\frac{\left(Tu-\left(s-k/2\right)\right)/T}{\widehat{b}_{2,T}\left(u\right)}\right)-K_2\left(\frac{\left(Tu-\left(s-k/2\right)\right)/T}{b_{\theta_2,T}\left(u\right)}\right) \tag{N.23}$$

$$\leq C_4\left|\frac{Tu-\left(s-k/2\right)}{T\widehat{b}_{2,T}\left(u\right)}-\frac{Tu-\left(s-k/2\right)}{Tb_{\theta_2,T}\left(u\right)}\right|$$

$$\leq CT^{-4/5-2/5}T^{2/5}\left|\left(\frac{\widehat{D}_2\left(u\right)}{\widehat{D}_1\left(u\right)}\right)^{-1/5}-\left(\frac{D_2\left(u\right)}{D_{1,\theta}\left(u\right)}\right)^{-1/5}\right|\left|Tu-\left(s-k/2\right)\right|$$

$$\leq CT^{-4/5-2/5}O_{\mathbb{P}}\left(1\right)\left|Tu-\left(s-k/2\right)\right|.$$

Therefore,

$$
\begin{aligned}
&T^{8q/10(2q+1)}\left(\widehat{J}_T\left(b_{\theta_1,T},\,\widehat{\bar{b}}_{2,T}\right)-\widehat{J}_T\left(b_{\theta_1,T},\,b_{\theta_2,T}\right)\right) \qquad \text{(N.24)}\\
&=T^{8q/10(2q+1)}\sum_{k=-T+1}^{T-1}K_1\left(b_{\theta_1,T}k\right)\frac{n_T}{T}\sum_{r=0}^{\lfloor T/n_T\rfloor}\left(\widehat{c}\left(rn_T/T,\,k\right)-\widehat{c}_{\theta_2,T}\left(rn_T/T,\,k\right)\right)\\
&\leq T^{8q/10(2q+1)}C\sum_{k=-T+1}^{T-1}\left|K_1\left(b_{\theta_1,T}k\right)\right|\frac{n_T}{T}\sum_{r=0}^{\lfloor T/n_T\rfloor}\frac{1}{T\bar{b}_{\theta_2,T}}\\
&\times\sum_{s=k+1}^{T}\left|K_2\left(\frac{\left(\left(r+1\right)n_T-\left(s-k/2\right)\right)/T}{\widehat{b}_{2,T}\left(\left(r+1\right)n_T/T\right)}\right)-K_2\left(\frac{\left(\left(r+1\right)n_T-\left(s-k/2\right)\right)/T}{b_{\theta_2,T}\left(\left(r+1\right)n_T/T\right)}\right)\right|\\
&\times\left|\left(\widehat{V}_s\widehat{V}_{s-k}-V_sV_{s-k}\right)+\left(V_sV_{s-k}-\mathbb{E}\left(V_sV_{s-k}\right)\right)+\mathbb{E}\left(V_sV_{s-k}\right)\right|\\
&\triangleq H_{1,T}+H_{2,T}+H_{3,T}.
\end{aligned}
$$

We have to show that $H_{1,T}+H_{2,T}+H_{3,T}\xrightarrow{\mathbb{P}}0$. Let $H_{1,1,T}$ (resp., $H_{1,2,T}$) be defined as $H_{1,T}$ but with the sum over k restricted to $k=1,\ldots,\,S_T$ (resp., $k=S_T+1,\ldots,\,T$). By a mean-value expansion, using (N.23),

$$
\begin{aligned}
\left|H_{1,1,T}\right|&\leq CT^{8q/10(2q+1)}T^{-1/2}\sum_{k=1}^{S_T}\left|K_1\left(b_{\theta_1,T}k\right)\right|\frac{n_T}{T}\sum_{r=0}^{\lfloor T/n_T\rfloor}\frac{1}{T\bar{b}_{\theta_2,T}}\\
&\sum_{s=k+1}^{T}\left|K_2\left(\frac{\left(\left(r+1\right)n_T-\left(s-k/2\right)\right)/T}{\widehat{\bar{b}}_{2,T}\left(\left(r+1\right)n_T/T\right)}\right)-K_2\left(\frac{\left(\left(r+1\right)n_T-\left(s-k/2\right)\right)/T}{b_{\theta_2,T}\left(\left(r+1\right)n_T/T\right)}\right)\right|\\
&\times\left\|V_s\left(\overline{\beta}\right)\frac{\partial}{\partial\beta}V_{s-k}\left(\overline{\beta}\right)+V_{s-k}\left(\overline{\beta}\right)\frac{\partial}{\partial\beta}V_s\left(\overline{\beta}\right)\right\|\sqrt{T}\left\|\widehat{\beta}-\beta_0\right\|\\
&\leq CT^{8q/10(2q+1)}\bar{b}_{\theta_2,T}^{-1}T^{-1/2-2/5}S_T\frac{n_T}{T}\sum_{r=0}^{\lfloor T/n_T\rfloor}O_{\mathbb{P}}\left(1\right)\\
&\times\left(\left(T^{-1}\sum_{s=1}^{T}\sup_{\beta\in\Theta}V_s^2\left(\beta\right)\right)^2\left(T^{-1}\sum_{s=1}^{T}\sup_{\beta\in\Theta}\left\|\frac{\partial}{\partial\beta}V_s\left(\beta\right)\right\|^2\right)^{1/2}\right)\sqrt{T}\left\|\widehat{\beta}-\beta_0\right\|.
\end{aligned}
$$

Using Assumption 3.3 the right-hand side above is such that

$$
CT^{8q/10(2q+1)}T^{-1/2-2/5}b_{\theta_2,T}^{-1}S_T\frac{n_T}{T}\sum_{r=0}^{\lfloor T/n_T\rfloor}O_{\mathbb{P}}\left(1\right)\xrightarrow{\mathbb{P}}0,
$$

since $r<7/8+3q/4$. Next,

$$
\begin{aligned}
\left|H_{1,2,T}\right|&\leq CT^{8q/10(2q+1)}T^{-1/2}\sum_{k=S_T+1}^{T-1}\left(b_{\theta_1,T}k\right)^{-b}\frac{n_T}{T}\sum_{r=0}^{\lfloor T/n_T\rfloor}\frac{1}{Tb_{\theta_2,T}}\\
&\times\sum_{s=k+1}^{T}\left|K_2\left(\frac{\left(\left(r+1\right)n_T-\left(s-k/2\right)\right)/T}{\widehat{b}_{2,T}\left(\left(r+1\right)n_T/T\right)}\right)-K_2\left(\frac{\left(\left(r+1\right)n_T-\left(s-k/2\right)\right)/T}{b_{\theta_2,T}\left(\left(r+1\right)n_T/T\right)}\right)\right|\\
&\times\left\|V_s\left(\overline{\beta}\right)\frac{\partial}{\partial\beta}V_{s-k}\left(\overline{\beta}\right)+V_{s-k}\left(\overline{\beta}\right)\frac{\partial}{\partial\beta}V_s\left(\overline{\beta}\right)\right\|\sqrt{T}\left\|\widehat{\beta}-\beta_0\right\|
\end{aligned}
$$

$$\leq CT^{8q/10(2q+1)} b_{\theta_2,T}^{-1} T^{-1/2-2/5} b_{\theta_1,T}^{-b} \sum_{k=S_T+1}^{T-1} k^{-b} \frac{n_T}{T} \sum_{r=0}^{\lfloor T/n_T \rfloor} O_{\mathbb{P}}(1)$$

$$\times \left(\left(T^{-1} \sum_{s=1}^{T} \sup_{\beta \in \Theta} V_s^2(\beta) \right)^2 \left(T^{-1} \sum_{s=1}^{T} \sup_{\beta \in \Theta} \left\| \frac{\partial}{\partial \beta} V_s(\beta) \right\|^2 \right)^{1/2} \right) \sqrt{T} \left\| \widehat{\beta} - \beta_0 \right\|$$

$$\leq CT^{8q/10(2q+1)} b_{\theta_2,T}^{-1} T^{-1/2-2/5} b_{\theta_1,T}^{-b} \sum_{k=S_T+1}^{T-1} k^{-b} O_{\mathbb{P}}(1)$$

$$\leq CT^{8q/10(2q+1)} b_{\theta_2,T}^{-1} T^{-1/2-2/5} b_{\theta_1,T}^{-b} S_T^{1-b} O_{\mathbb{P}}(1)$$

$$\leq CT^{8q/10(2q+1)} b_{\theta_2,T}^{-1} T^{-1/2-2/5} b_{\theta_1,T}^{-b} b_{\theta_1,T}^{-r(1-b)} O_{\mathbb{P}}(1)$$

$$\leq CT^{8q/10(2q+1)} b_{\theta_2,T}^{-1} T^{-1/2-2/5} b_{\theta_1,T}^{-b} T^{4r(1-b)/5(2q+1)} O_{\mathbb{P}}(1) \to 0,$$

since $r > (8b - 7 - 6q)/8(b-1)$. This shows $H_{1,T} \xrightarrow{\mathbb{P}} 0$. Let $H_{2,1,T}$ (resp., $H_{2,2,T}$) be defined as $H_{2,T}$ but with the sum over k restricted to $k = 1, \ldots, S_T$ (resp., $k = S_T + 1, \ldots, T$). We have

$$\mathbb{E}\left(H_{2,1,T}^2\right) \leq T^{8q/5(2q+1)} \sum_{k=1}^{S_T} \sum_{j=1}^{S_T} K_1(b_{\theta_1,T} k) K_1(b_{\theta_1,T} j) \left(\frac{n_T}{T}\right)^2 \sum_{r_1=0}^{\lfloor T/n_T \rfloor} \sum_{r_2=0}^{\lfloor T/n_T \rfloor} \frac{1}{(T b_{\theta_2,T})^2} \tag{N.25}$$

$$\times \sum_{s=k+1}^{T} \sum_{t=j+1}^{T} \left| K_2\left(\frac{((r_1+1) n_T - (s - k/2))/T}{\widehat{b}_{2,T}((r_1+1) n_T/T)} \right) - K_2\left(\frac{((r_1+1) n_T - (s - k/2))/T}{b_{\theta_2,T}((r_1+1) n_T/T)} \right) \right|$$

$$\times \left| K_2\left(\frac{((r_2+1) n_T - (t - j/2))/T}{\widehat{b}_{2,T}((r_2+1) n_T/T)} \right) - K_2\left(\frac{((r_2+1) n_T - (t - j/2))/T}{b_{\theta_2,T}((r_2+1) n_T/T)} \right) \right|$$

$$\times \left| \mathbb{E}\left(V_s V_{s-k} - \mathbb{E}(V_s V_{s-k}) \right)\left(V_t V_{t-k} - \mathbb{E}(V_t V_{t-k}) \right) \right|$$

$$\leq CT^{8q/5(2q+1)} S_T^2 T^{-2/5} (T b_{\theta_2,T})^{-1} \sup_{k \geq 1} T b_{\theta_2,T} \mathrm{Var}\left(\widetilde{\Gamma}(k)\right) O_{\mathbb{P}}(1)$$

$$\leq CT^{(8q+8r)/5(2q+1)-2/5-1} O_{\mathbb{P}}\left(b_{\theta_2,T}^{-1}\right) \to 0,$$

where we have used Lemma S.A.5 and $r < (6 + 4q)/8$. Turning to $H_{2,2,T}$,

$$\mathbb{E}\left(H_{2,2,T}^2\right) \leq T^{8q/5(2q+1)-2/5} (T b_{\theta_2,T})^{-1} b_{\theta_1,T}^{-2b} \left(\sum_{k=S_T+1}^{T-1} k^{-b} \sqrt{T b_{\theta_2,T}} \left(\mathrm{Var}\left(\widetilde{\Gamma}(k)\right)\right)^{1/2} O(1) \right)^2 \tag{N.26}$$

$$\leq T^{8q/5(2q+1)} T^{-2/5-1} b_{\theta_2,T}^{-1} b_{\theta_1,T}^{-2b} \left(\sum_{k=S_T+1}^{T-1} k^{-b} \sqrt{T b_{\theta_2,T}} \left(\mathrm{Var}\left(\widetilde{\Gamma}(k)\right)\right)^{1/2} \right)^2$$

$$\leq T^{8q/5(2q+1)} T^{-2/5-1} b_{\theta_2,T}^{-1} b_{\theta_1,T}^{-2b} \left(\sum_{k=S_T+1}^{T-1} k^{-b} O(1) \right)^2$$

$$\leq T^{8q/5(2q+1)} T^{-2/5-1} b_{\theta_2,T}^{-1} b_{\theta_1,T}^{-2b} S_T^{2(1-b)} \to 0,$$

since $r > (b - 3/4 - q/2)/(b-1)$. Combining (N.25)-(N.26) yields $H_{2,T} \xrightarrow{\mathbb{P}} 0$. Let $H_{3,1,T}$ (resp., $H_{3,2,T}$) be defined as $H_{3,T}$ but with the sum over k restricted to $k = 1, \ldots, S_T$ (resp., $k = S_T + 1, \ldots, T$). Given

$|K_1(\cdot)| \leq 1$ and (N.23), we have

$$|H_{3,1,T}| \leq CT^{8q/10(2q+1)} T^{-2/5} \sum_{k=1}^{S_T} |\Gamma_T(k)|$$
$$\leq CT^{8q/10(2q+1)} T^{-2/5} \sum_{k=1}^{\infty} k^{-l} \to 0,$$

since $\sum_{k=1}^{\infty} k^{-l} < \infty$ for $l > 1$ and $T^{8q/10(2q+1)} T^{-2/5} \to 0$. Finally,

$$|H_{3,2,T}| \leq CT^{8q/10(2q+1)} T^{-2/5} \sum_{k=S_T+1}^{T-1} |\Gamma_T(k)|$$
$$\leq CT^{8q/10(2q+1)} T^{-2/5} \sum_{k=S_T+1}^{T-1} k^{-l}$$
$$\leq CT^{8q/10(2q+1)} T^{-2/5} S_T^{1-l}$$
$$\leq CT^{8q/10(2q+1)} T^{-2/5} T^{4r(1-l)/5(2q+1)} \to 0.$$

This completes the proof of part (ii).

We move to part (i). For some arbitrary $\phi_{\theta^*} \in (0, \infty)$, $\widehat{J}_T(b_{\theta_1,T}, b_{\theta_2,T}) - J_T = o_{\mathbb{P}}(1)$ by Theorem 3.2-(i) since $b_{\theta_2,T} = O(T^{-1/5})$ and $q > 1/2$ imply that $\sqrt{T} b_{1,T} \to \infty$ holds. Hence, it remains to show $\widehat{J}_T(b_{\theta_1,T}, b_{\theta_2,T}) - \widehat{J}_T(\widehat{b}_{1,T}, \widehat{b}_{2,T}) = o_{\mathbb{P}}(1)$. This result differs from that of part (ii) only because the scale factor $T^{8q/10(2q+1)}$ does not appear, Assumption 5.1-(ii) is replaced by part (i) of the same assumption, Assumption 5.1-(iii, v, vi) is not imposed, and $q > 1/2$. Let S_T be defined as in part (ii) and

$$r \in (\max\{(8b - 10q - 5)/8(b-1), (b - 1/2 - q)/(b-1)\},$$
$$\min\{13/16 + 5q/8, (3+2q)/4, 1\}),$$

with $b > 1 + 1/q$. We will use the decomposition in (N.8), and N_1 and N_2 as defined after (N.8). Let $A_{1,T}$, $A_{2,T}$ and $A_{3,T}$ be as in (N.9) without the scale factor $T^{8q/10(2q+1)}$. Proceeding as in (N.10), we have

$$|A_{1,1,T}| \leq \sum_{k=1}^{S_T} C_2 \left|\widehat{b}_{1,T} - b_{\theta_1,T}\right| k \left|\widehat{\Gamma}(k)\right| \tag{N.27}$$
$$\leq C \left|\widehat{\phi}(q)^{1/(2q+1)} - \phi_{\theta^*}^{1/(2q+1)}\right| \left(\widehat{\phi}(q)\,\phi_{\theta^*}\right)^{-1/(2q+1)} \left(T\widehat{\overline{b}}_{2,T}\right)^{-1/(2q+1)} \sum_{k=1}^{S_T} k \left|\widehat{\Gamma}(k)\right|,$$

for some $C < \infty$. By Assumption 5.1-(i),

$$\left|\widehat{\phi}(q)^{1/(2q+1)} - \phi_{\theta^*}^{1/(2q+1)}\right| \left(\widehat{\phi}(q)\,\phi_{\theta^*}\right)^{-1/(2q+1)} = O_{\mathbb{P}}(1).$$

Then, it suffices to show that $B_{1,T} + B_{2,T} + B_{3,T} \xrightarrow{\mathbb{P}} 0$, where

$$B_{1,T} = \left(T\widehat{\overline{b}}_{2,T}\right)^{-1/(2q+1)} \sum_{k=1}^{S_T} k \left|\widehat{\Gamma}(k) - \widetilde{\Gamma}(k)\right|, \tag{N.28}$$
$$B_{2,T} = \left(T\widehat{\overline{b}}_{2,T}\right)^{-1/(2q+1)} \sum_{k=1}^{S_T} k \left|\widetilde{\Gamma}(k) - \Gamma_T(k)\right|,$$

$$B_{3,T}=\left(T\widehat{\overline{b}}_{2,T}\right)^{-1/(2q+1)}\sum_{k=1}^{S_T}k\left|\Gamma_T\left(k\right)\right|.$$

By a mean-value expansion, we have

$$\begin{aligned}B_{1,T}&\leq\left(T\widehat{\overline{b}}_{2,T}\right)^{-1/(2q+1)}T^{-1/2}\sum_{k=1}^{S_T}k\left|\left(\frac{\partial}{\partial\beta'}\widehat{\Gamma}\left(k\right)|_{\beta=\overline{\beta}}\right)\sqrt{T}\left(\widehat{\beta}-\beta_0\right)\right| \\ &\leq C\left(T\widehat{\overline{b}}_{2,T}\right)^{-1/(2q+1)}\left(Tb_{\theta_2,T}\right)^{2r/(2q+1)}T^{-1/2}\sup_{k\geq1}\left\|\frac{\partial}{\partial\beta}\widehat{\Gamma}\left(k\right)|_{\beta=\overline{\beta}}\right\|\sqrt{T}\left\|\widehat{\beta}-\beta_0\right\|,\end{aligned}\tag{N.29}$$

since $r<13/16+5q/8$, and $\sup_{k\geq1}||\left(\partial/\partial\beta\right)\widehat{\Gamma}\left(k\right)|_{\beta=\overline{\beta}}||=O_{\mathbb{P}}\left(1\right)$ using (S.27) and Assumption 3.3-(ii,iii). In addition,

$$\begin{aligned}\mathbb{E}\left(B_{2,T}^2\right)&\leq\mathbb{E}\left(\left(T\widehat{\overline{b}}_{2,T}\right)^{-2/(2q+1)}\sum_{k=1}^{S_T}\sum_{j=1}^{S_T}kj\left|\widetilde{\Gamma}\left(k\right)-\Gamma_T\left(k\right)\right|\left|\widetilde{\Gamma}\left(j\right)-\Gamma_T\left(j\right)\right|\right) \\ &\leq\mathbb{E}\left(\left(T\widehat{\overline{b}}_{2,T}\right)^{-2/(2q+1)}\sum_{k=1}^{S_T}\sum_{j=1}^{S_T}kj\left|\widetilde{\Gamma}\left(k\right)-\Gamma_T\left(k\right)\right|\left|\widetilde{\Gamma}\left(j\right)-\Gamma_T\left(j\right)\right|\right) \\ &\leq\left(T\widehat{\overline{b}}_{2,T}\right)^{-2/(2q+1)-1}S_T^4\sup_{k\geq1}T\widehat{\overline{b}}_{2,T}\mathrm{Var}\left(\widetilde{\Gamma}\left(k\right)\right) \\ &\leq\left(T\widehat{\overline{b}}_{2,T}\right)^{-2/(2q+1)-1}\left(Tb_{2,T}\right)^{4r/(2q+1)}\sup_{k\geq1}T\widehat{\overline{b}}_{2,T}\mathrm{Var}\left(\widetilde{\Gamma}\left(k\right)\right) \\ &\leq\widehat{\overline{b}}_{2,T}^{-2/(2q+1)-1}T^{-1-2/(2q+1)}T^{16r/5(2q+1)}\sup_{k\geq1}T\widehat{\overline{b}}_{2,T}\mathrm{Var}\left(\widetilde{\Gamma}\left(k\right)\right)\rightarrow0,\end{aligned}\tag{N.30}$$

given that $\sup_{k\geq1}T\widehat{\overline{b}}_{2,T}\mathrm{Var}(\widetilde{\Gamma}\left(k\right))=O\left(1\right)$ by Lemma S.A.5 and $r<\left(3+2q\right)/4$. The bound in (N.14) is replaced by

$$\begin{aligned}B_{3,T}&\leq\left(T\widehat{\overline{b}}_{2,T}\right)^{-1/(2q+1)}S_T\sum_{k=1}^{\infty}\left|\Gamma_T\left(k\right)\right| \\ &\leq\left(T\widehat{\overline{b}}_{2,T}\right)^{(r-1)/(2q+1)}O_{\mathbb{P}}\left(1\right)\rightarrow0,\end{aligned}\tag{N.31}$$

using Assumption 3.2-(i) since $r<1$. This gives $A_{1,T}\xrightarrow{\mathbb{P}}0$. Next, we show that $A_{2,T}\xrightarrow{\mathbb{P}}0$. As above, let $A_{2,1,T}=L_{1,T}+L_{2,T}+L_{3,T}$ where each summand is defined as in (N.15) without the factor $T^{8q/10(2q+1)}$. Equation (N.16) is then replaced by

$$\begin{aligned}\left|L_{1,T}\right|&=T^{-1/2}\sum_{k=S_T+1}^{T-1}C_1\left(\widehat{b}_{1,T}k\right)^{-b}\left|\left(\frac{\partial}{\partial\beta'}\widehat{\Gamma}\left(k\right)\right)|_{\beta=\overline{\beta}}\sqrt{T}\left(\widehat{\beta}-\beta_0\right)\right| \\ &=T^{-1/2+4b/5(2q+1)}\sum_{k=S_T+1}^{T-1}C_1k^{-b}\left|\left(\frac{\partial}{\partial\beta'}\widehat{\Gamma}\left(k\right)\right)|_{\beta=\overline{\beta}}\sqrt{T}\left(\widehat{\beta}-\beta_0\right)\right| \\ &=T^{-1/2+4b/5(2q+1)+4r(1-b)/5(2q+1)}\left|\left(\frac{\partial}{\partial\beta'}\widehat{\Gamma}\left(k\right)\right)|_{\beta=\overline{\beta}}\sqrt{T}\left(\widehat{\beta}-\beta_0\right)\right| \\ &=T^{-1/2+4b/5(2q+1)+4r(1-b)/5(2q+1)}O\left(1\right)O_{\mathbb{P}}\left(1\right),\end{aligned}\tag{N.32}$$

which converges to zero since $r > (8b - 10q - 5)/8(b-1)$. Also, (N.17) is replaced by

$$\begin{aligned} |L_{2,T}| &= \sum_{k=S_T+1}^{T-1} C_1 \left(\widehat{b}_{1,T} k\right)^{-b} \left|\widetilde{\Gamma}(k) - \Gamma_T(k)\right| \\ &= C_1 \left(q K_{1,q}^2 \widehat{\phi}(q)\right)^{b/(2q+1)} T^{b/(2q+1)-1/2} \widehat{b}_{2,T}^{b/(2q+1)-1/2} \left(\sum_{k=S_T+1}^{T-1} k^{-b}\right) \sqrt{T\widehat{b}_{2,T}} \left|\widetilde{\Gamma}(k) - \Gamma_T(k)\right|, \end{aligned} \tag{N.33}$$

and the bound in (N.18) is replaced by,

$$\begin{aligned} &\mathbb{E}\left(T^{b/(2q+1)-1/2} \widehat{\bar{b}}_{2,T}^{b/(2q+1)-1/2} \sum_{k=S_T}^{T-1} k^{-b} \sqrt{T\widehat{\bar{b}}_{2,T}} \left|\widetilde{\Gamma}(k) - \Gamma_T(k)\right|\right)^2 \\ &\leq T^{2b/(2q+1)-1} \widehat{\bar{b}}_{2,T}^{2b/(2q+1)-1} \left(\sum_{k=S_T}^{T-1} k^{-b} \sqrt{T\widehat{\bar{b}}_{2,T}} \left(\mathrm{Var}\left(\widetilde{\Gamma}(k)\right)\right)^{1/2}\right)^2 \\ &= T^{2b/(2q+1)-1} \widehat{\bar{b}}_{2,T}^{2b/(2q+1)-1} \left(\sum_{k=S_T}^{T-1} k^{-b}\right)^2 O(1) \\ &= T^{2b/(2q+1)-1} \widehat{\bar{b}}_{2,T}^{2b/(2q+1)-1} S_T^{2(1-b)} O(1) \to 0, \end{aligned} \tag{N.34}$$

since $r > (b - 1/2 - q)/(b-1)$ and $Tb_{2,T}\mathrm{Var}(\widetilde{\Gamma}(k)) = O(1)$, as above. Combining (N.33)-(N.34) yields $L_{2,T} \xrightarrow{\mathbb{P}} 0$ since $\widehat{\phi}(q) = O_{\mathbb{P}}(1)$. Let us turn to $L_{3,T}$. We have (N.21) replaced by,

$$\begin{aligned} \left|\sum_{k=S_T+1}^{T-1} K_1\left(\widehat{b}_{1,T} k\right) \Gamma_T(k)\right| &\leq \sum_{k=S_T+1}^{T-1} \frac{n_T}{T} \sum_{r=0}^{\lfloor T/n_T \rfloor} |c(rn_T/T, k)| \\ &\leq \sum_{k=S_T+1}^{T-1} \sup_{u \in [0,1]} |c(u, k)| \to 0. \end{aligned} \tag{N.35}$$

Equations (N.32)-(N.35) imply $A_{2,1,T} \xrightarrow{\mathbb{P}} 0$. Thus, as in the proof of part (ii), we have $A_{2,T} \xrightarrow{\mathbb{P}} 0$ and $A_{3,T} \xrightarrow{\mathbb{P}} 0$. It remains to show that $(\widehat{J}_T(b_{\theta_1,T}, \widehat{\bar{b}}_{2,T}) - \widehat{J}_T(b_{\theta_1,T}, \bar{b}_{\theta_2,T})) \xrightarrow{\mathbb{P}} 0$. Let $\widehat{c}_{\theta_2,T}(rn_T/T, k)$ be defined as in part (ii). We have (N.22), and (N.23) is replaced by

$$\begin{aligned} &K_2\left(\frac{(Tu - (s-k/2)/T)}{\widehat{b}_{2,T}(u)}\right) - K_2\left(\frac{(Tu - (s-k/2)/T)}{b_{\theta_2,T}(u)}\right) \\ &\leq C_4 \left|\frac{Tu - (s-k/2)}{T\widehat{b}_{2,T}(u)} - \frac{Tu - (s-k/2)}{Tb_{\theta_2,T}(u)}\right| \\ &\leq C_4 T^{-1} \left|\frac{Tu - (s-k/2)\left(\widehat{b}_{2,T}(u) - b_{\theta_2,T}(u)\right)}{\widehat{b}_{2,T}(u)\, b_{\theta_2,T}(u)}\right| \\ &= C_4 T^{-4/5} \left(\left(\frac{\widehat{D}_1(u)}{\widehat{D}_2(u)}\right)\left(\frac{D_{1,\theta}(u)}{D_2(u)}\right)\right)^{1/5} \left|\left(\frac{\widehat{D}_2(u)}{\widehat{D}_1(u)}\right)^{1/5} - \left(\frac{D_2(u)}{D_{1,\theta}(u)}\right)^{1/5}\right| |Tu - (s-k/2)| \\ &= CT^{-4/5} |Tu - (s-k/2)|, \end{aligned} \tag{N.36}$$

for $s\in\{Tu-\lfloor Tb_{\theta_2,T}(u)\rfloor,\ldots,Tu+\lfloor Tb_{\theta_2,T}(u)\rfloor\}$, where $u=(r+1)\,n_T/T$. Therefore,

$$
\begin{aligned}
&\widehat{J}_T\left(b_{\theta_1,T},\,\widehat{\bar{b}}_{2,T}\right)-\widehat{J}_T\left(b_{\theta_1,T},\,\bar{b}_{\theta_2,T}\right)\\
&\quad=\sum_{k=-T+1}^{T-1}K_1\left(b_{\theta_1,T}k\right)\frac{n_T}{T}\sum_{r=0}^{\lfloor T/n_T\rfloor}\left(\widehat{c}\left(rn_T/T,\,k\right)-\widehat{c}_{\theta_2,T}\left(rn_T/T,\,k\right)\right)\\
&\quad\leq C\sum_{k=-T+1}^{T-1}K_1\left(b_{\theta_1,T}k\right)\\
&\quad\times\frac{n_T}{T}\sum_{r=0}^{\lfloor T/n_T\rfloor}\frac{1}{T\bar{b}_{\theta_2,T}}\sum_{s=k+1}^{T}\left|K_2\left(\frac{((r+1)\,n_T-(s-k/2))\,/T}{\widehat{b}_{2,T}\left((r+1)\,n_T/T\right)}\right)-K_2\left(\frac{((r+1)\,n_T-(s-k/2))\,/T}{b_{\theta_2,T}\left((r+1)\,n_T/T\right)}\right)\right|\\
&\quad\times\left|\left(\widehat{V}_s\widehat{V}_{s-k}-V_sV_{s-k}\right)+\left(V_sV_{s-k}-\mathbb{E}\left(V_sV_{s-k}\right)\right)+\mathbb{E}\left(V_sV_{s-k}\right)\right|\\
&\quad\triangleq H_{1,T}+H_{2,T}+H_{3,T}.
\end{aligned}
$$

We have to show that $H_{1,T}+H_{2,T}+H_{3,T}\xrightarrow{\mathbb{P}}0$. By a mean-value expansion, using (N.36),

$$
\begin{aligned}
\left|H_{1,T}\right|&\leq CT^{-1/2}\sum_{k=-T+1}^{T-1}\left|K_1\left(b_{\theta_1,T}k\right)\right|\\
&\quad\times\frac{n_T}{T}\sum_{r=0}^{\lfloor T/n_T\rfloor}\frac{1}{Tb_{\theta_2,T}}\sum_{s=k+1}^{T}\left|K_2\left(\frac{((r+1)\,n_T-(s-k/2))\,/T}{\widehat{b}_{2,T}\left((r+1)\,n_T/T\right)}\right)-K_2\left(\frac{((r+1)\,n_T-(s-k/2))\,/T}{b_{\theta_2,T}\left((r+1)\,n_T/T\right)}\right)\right|\\
&\quad\times\left\|V_s\left(\overline{\beta}\right)\frac{\partial}{\partial\beta}V_{s-k}\left(\overline{\beta}\right)+V_{s-k}\left(\overline{\beta}\right)\frac{\partial}{\partial\beta}V_s\left(\overline{\beta}\right)\right\|\sqrt{T}\left\|\widehat{\beta}-\beta_0\right\|\\
&\leq Cb_{\theta_2,T}^{-1}T^{-1/2}\sum_{k=-T+1}^{T-1}\left|K_1\left(b_{\theta_1,T}k\right)\right|\\
&\quad\times\frac{n_T}{T}\sum_{r=0}^{\lfloor T/n_T\rfloor}CO_{\mathbb{P}}(1)\\
&\quad\times\left(\left(T^{-1}\sum_{s=1}^{T}\sup_{\beta\in\Theta}V_s^2\left(\beta\right)\right)^2\left(T^{-1}\sum_{s=1}^{T}\sup_{\beta\in\Theta}\left\|\frac{\partial}{\partial\beta}V_s\left(\beta\right)\right\|^2\right)^{1/2}\right)\sqrt{T}\left\|\widehat{\beta}-\beta_0\right\|.
\end{aligned}
$$

Using Assumption 3.3 and (N.36), the right-hand side above is such that

$$
CT^{-1/2}b_{\theta_1,T}^{-1}b_{\theta_2,T}^{-1}b_{\theta_1,T}\sum_{k=-T+1}^{T-1}\left|K_1\left(b_{\theta_1,T}k\right)\right|\frac{n_T}{T}\sum_{r=0}^{\lfloor T/n_T\rfloor}CO_{\mathbb{P}}(1)\xrightarrow{\mathbb{P}}0,
$$

since $T^{-1/2}b_{\theta_1,T}^{-1}b_{\theta_2,T}^{-1}\to0$. This shows $H_{1,T}\xrightarrow{\mathbb{P}}0$. Let $H_{2,1,T}$ (resp. $H_{2,2,T}$) be defined as $H_{2,T}$ but with the sum over k restricted to $k=1,\ldots,S_T$ (resp., $k=S_T+1,\ldots,T$). We have

$$
\begin{aligned}
\mathbb{E}\left(H_{2,1,T}^2\right)&\leq\sum_{k=1}^{S_T}\sum_{j=1}^{S_T}K_1\left(b_{\theta_1,T}k\right)K_1\left(b_{\theta_1,T}j\right)\\
&\quad\times\left(\frac{n_T}{T}\right)^2\sum_{r_1=0}^{\lfloor T/n_T\rfloor}\sum_{r_2=0}^{\lfloor T/n_T\rfloor}\frac{1}{\left(Tb_{\theta_2,T}\right)^2}\sum_{s=k+1}^{T}\sum_{t=j+1}^{T}
\end{aligned}
\tag{N.37}
$$

$$
\begin{aligned}
&\times\left|K_2\left(\frac{((r_1+1)\,n_T-(s-k/2))\,/T}{\widehat{b}_{2,T}\,((r_1+1)\,n_T/T)}\right)-K_2\left(\frac{((r_1+1)\,n_T-(s-k/2))\,/T}{b_{\theta_2,T}\,((r_1+1)\,n_T/T)}\right)\right|\\
&\times\left|K_2\left(\frac{((r_2+1)\,n_T-(t-j/2))\,/T}{\widehat{b}_{2,T}\,((r_2+1)\,n_T/T)}\right)-K_2\left(\frac{((r_2+1)\,n_T-(t-j/2))\,/T}{b_{\theta_2,T}\,((r_2+1)\,n_T/T)}\right)\right|\\
&\times\left|V_sV_{s-k}-\mathbb{E}\left(V_sV_{s-k}\right)\left(V_tV_{t-k}-\mathbb{E}\left(V_tV_{t-k}\right)\right)\right|.\\
&\leq CS_T^2\left(Tb_{\theta_2,T}\right)^{-1}\sup_{k\geq 1}Tb_{\theta_2,T}\mathrm{Var}\left(\widetilde{\Gamma}\left(k\right)\right)O_{\mathbb{P}}\left(1\right)\\
&\leq CT^{8r/5(2q+1)}O_{\mathbb{P}}\left(T^{-1}b_{\theta_2,T}^{-1}\right)\to 0,
\end{aligned}
$$

where we have used Lemma S.A.5, (N.36) and $r<3/2$. Turning to $H_{2,2,T}$,

$$
\begin{aligned}
\mathbb{E}\left(H_{2,2,T}^2\right)&\leq\left(Tb_{\theta_2,T}\right)^{-1}b_{\theta_1,T}^{-2b}\left(\sum_{k=S_T+1}^{T-1}k^{-b}\sqrt{Tb_{\theta_2,T}}\left(\mathrm{Var}\left(\widetilde{\Gamma}\left(k\right)\right)\right)^{1/2}O\left(1\right)\right)^2 \qquad \text{(N.38)}\\
&\leq T^{-1}b_{\theta_2,T}^{-1}b_{\theta_1,T}^{-2b}\left(\sum_{k=S_T+1}^{T-1}k^{-b}\sqrt{Tb_{\theta_2,T}}\left(\mathrm{Var}\left(\widetilde{\Gamma}\left(k\right)\right)\right)^{1/2}\right)^2\\
&\leq T^{-1}b_{\theta_2,T}^{-1}b_{\theta_1,T}^{-2b}\left(\sum_{k=S_T+1}^{T-1}k^{-b}O\left(1\right)\right)^2\\
&\leq T^{-1}b_{\theta_2,T}^{-1}b_{\theta_1,T}^{-2b}S_T^{2(1-b)}\to 0,
\end{aligned}
$$

since $r>\left(b-q-1/2\right)/\left(b-1\right)$. Combining (N.37)-(N.38) yields $H_{2,T}\overset{\mathbb{P}}{\to}0$. Given $\left|K_1\left(\cdot\right)\right|\leq 1$ and (N.36), we have $\left|H_{3,T}\right|\leq C\sum_{k=-\infty}^{\infty}\left|\Gamma_T\left(k\right)\right|o_{\mathbb{P}}\left(1\right)\to 0$. This concludes the proof of part (i).

The result of part (iii) follows from the same argument as in Theorem 3.2-(iii) with references to Theorem 3.2-(i,ii) changed to Theorem 5.1-(i,ii). □

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