Theory of Low Frequency Contamination from Nonstationarity and Misspecification: Consequences for HAR Inference

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4th March 2021

Abstract

We establish theoretical results about the low frequency contamination induced by general nonstationarity for estimates such as the sample autocovariance and the periodogram, and deduce consequences for heteroskedasticity and autocorrelation robust (HAR) inference. We show that for short memory nonstationarity data these estimates exhibit features akin to long memory. We present explicit expressions for the asymptotic bias of these estimates. This bias increases with the degree of heterogeneity. The sample autocovariances display hyperbolic rather than exponential decay while the periodogram becomes unbounded near the origin. We distinguish cases where this contamination only occurs as a small-sample problem and cases where the contamination continues to hold asymptotically. We show theoretically that nonparametric smoothing over time is robust to low frequency contamination. Simulations confirm that our theory provides useful approximations. Since the autocovariances and the periodogram are key elements for HAR inference, our results provide new insights on the debate between consistent versus inconsistent long-run variance (LRV) estimation. Existing LRV estimators tend to be inflated when the data are nonstationary. This results in HAR tests that can be undersized and exhibit dramatic power losses. These issues can potentially arise in most of the HAR inference contexts in econometrics. Our theory indicates that long bandwidths or fixed-b HAR tests suffer more from low frequency contamination relative to HAR tests based on HAC estimators, whereas recently introduced double kernel HAC estimators do not suffer from this problem.

JEL Classification: C12, C13, C18, C22, C32, C51

Keywords: Fixed-b, HAC standard errors, HAR, Long memory, Long-run variance, Low frequency contamination, Nonstationarity, Misspecification, Outliers, Segmented locally stationary.

*This paper is based on some parts of Chapter 1 of the first author’s doctoral dissertation at Boston University. We thank Whitney Newey and Tim Vogelsang for discussions. We also thank Adam McCloskey and Zhongjun Qu for comments.
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1 Introduction

Economic and financial time series are highly nonstationary [see, e.g., Perron (1989), Stock and Watson (1996), Ng and Wright (2013), and Giacomini and Rossi (2015)]. We develop theoretical results about the behavior of the sample autocovariance ($\hat{\Gamma}(k), k \in \mathbb{Z}$) and the periodogram ($I_T(\omega), \omega \in [-\pi, \pi]$) for a short memory nonstationary process.\footnote{By short memory nonstationary processes we mean processes that have non-constant moments and whose sum of absolute autocovariances is finite. The latter rules out processes with unbounded second moments (e.g., unit root). For unit root or trending time series, one has to first apply some differencing or de-trending technique.} We show that nonstationarity (e.g., time-variation in the mean or autocovariances) induces low frequency contamination, meaning that the sample autocovariance and the periodogram share features that are similar to those of a long memory series. We present explicit expressions for the asymptotic bias of these estimates, showing that it is always positive and increases with the degree of heterogeneity in the data.

The low frequency contamination can be explained as follows. For a short memory series, the autocorrelation function (ACF) is known to display exponential decay and to vanish as the lag length $k \to \infty$, and the periodogram is known to be finite at the origin. Under general forms of nonstationarity, we show theoretically that $\hat{\Gamma}(k) = \lim_{T \to \infty} \Gamma_T(k) + d^*$, where $\Gamma_T(k) = T^{-1} \sum_{k+1}^T \mathbb{E}(V_t V_{t-k}), k \geq 0$ and $d^* > 0$ is independent of $k$. Assuming positive dependence for simplicity (i.e., $\lim_{T \to \infty} \Gamma_T(k) > 0$), this means that each sample autocovariance overestimates the true dependence in the data. The bias factor $d^* > 0$ depends on the type of nonstationarity. Interestingly, $d^*$ in general does not vanish as $T \to \infty$. This yields a slower than exponential decay. In addition, since short memory implies $\Gamma_T(k) \to 0$ as $k \to \infty$, it follows that $d^*$ generates long memory effects since $\hat{\Gamma}(k) \approx d^* > 0$ as $k \to \infty$. As for the periodogram, $I_T(\omega)$, we show that under nonstationarity $\mathbb{E}(I_T(\omega)) \to \infty$ as $\omega \to 0$, a feature shared by long memory processes.

This low frequency contamination holds as an asymptotic approximation. We verify analytically the quality of the approximation to finite-sample situations. We distinguish cases where this contamination only occurs as a small-sample problem and cases where it continues to hold asymptotically. The former involves $d^* \approx 0$ asymptotically but a consistent estimate of $d^*$ satisfies $\hat{d}^* > 0$ in finite-sample. We show analytically that using $\hat{d}^*$ in place of $d^*$ provides a good approximation in this context. This helps us to explain the long memory effects in situations where asymptotically no low frequency contamination should occur. An example is a $t$-test on a regression coefficient in a correctly specified model with nonstationary errors that are serially correlated. Other examples include $t$-tests in the linear model with mild forms of misspecification that do not undermine the conditions for consistency of the least-squares estimator. Further, similar issues arise if one applies some prior de-trending techniques where the fitted model is not correctly specified (e.g., the data follow a nonlinear trend but one removes a linear trend). Yet another example for which our results are relevant is the case of outliers. In all these examples, $d^* \approx 0$ asymptotically but one observes enhanced persistence in finite-sample that can affect the properties of heteroskedasticity.
and autocorrelation robust (HAR) inference. Most of the HAR inference in applied work (besides the t- and F-test in regression models) are characterized by nonstationary alternative hypotheses for which \( d^* > 0 \) even asymptotically. This class of tests is very large. Tests for forecast evaluation [e.g., Casini (2018), Diebold and Mariano (1995), Giacomini and Rossi (2009, 2010), Giacomini and White (2006), Perron and Yamamoto (2021) and West (1996)], tests and inference for structural change models [e.g., Andrews (1993), Bai and Perron (1998), Casini and Perron (2020b, 2020c, 2020d), Elliott and Müller (2007), and Qu and Perron (2007)], tests and inference in time-varying parameters models [e.g., Cai (2007) and Chen and Hong (2012)], tests and inference for regime switching models [e.g., Hamilton (1989) and Qu and Zhuo (2020)] and others are part of this class.

We propose a solution to these problems via nonparametric smoothing over time which we show theoretically to be robust to low frequency contamination. By applying nonparametric smoothing, we prove that the sample local autocovariance and the local periodogram do not exhibit long memory features. Nonparametric smoothing avoids mixing highly heterogeneous data coming from distinct nonstationary regimes as opposed to what the sample autocovariance and the periodogram do. It was introduced recently for robust inference under nonstationarity by Casini (2021). He proposed a new HAC estimator that applies nonparametric smoothing over time in order to account flexibly for nonstationarity. Our results provide a theoretical justification for his double-kernel heteroskedasticity and autocorrelation consistent (DK-HAC) estimators.

Our work is different from the literature on spurious persistence caused by the presence of level shifts or other deterministic trends. Perron (1990) showed two useful results: first, the presence of breaks in mean often induces spurious non-rejection of the unit root hypothesis; second, the presence of a level shift asymptotically biases the estimate of the AR coefficient towards one. Bhattacharya, Gupta, and Waymire (1983) demonstrated that certain deterministic trends can induce the spurious presence of long memory. Via simulations, Lamoureux and Lastrapes (1990), and analytically, Hillebrand (2005), showed that when the mean of a GARCH process changes, the sum of the estimated AR parameters of the conditional variance converges in probability to one. Christensen and Varneskov (2017) and McCloskey and Hill (2017) provided methods to estimate the parameters of, respectively, fractional cointegrating vector and a stationary ergodic time series, robust to some forms of mean/trend changes. Diebold and Inoue (2001), Granger and Hyung (2004), Mikosch and Stărică (2004) and Perron and Qu (2010) showed via simulations and theoretical arguments that changes in mean induce hyperbolically decaying autocorrelations and spectral density estimates that approach infinity at the null frequency. Our results are different from theirs in that we consider a more general problem and we allow for more general forms of nonstationarity using the segmented locally stationary framework of Casini (2021). Importantly, we provide a general solution to these problems and show theoretically its robustness to low frequency contamination. Finally, we discuss in detail the implications of our theory for HAR inference.

HAR inference relies on estimation of the long-run variance (LRV). The latter, from a time
domain perspective, is equivalent to the sum of all autocovariances of some relevant process while, from a frequency domain perspective, is equal to $2\pi$ times an integrated time-varying spectral density at the zero frequency. From a time domain perspective, estimation involves a weighted sum of the sample autocovariances, while from a frequency domain perspective estimation is based on a weighted sum of the periodogram ordinates near the zero frequency. Therefore, our results on low frequency contamination for the sample autocovariances and the periodogram can have important implications in this context.

There are two main approaches in HAR inference which relies on whether the LRV estimator is consistent or not. The classical approach relies on consistency which results in HAC standard errors [cf. Newey and West (1987; 1994) and Andrews (1991)]. Inference is standard because HAR tests follow asymptotically standard distributions. It was shown early that HAC standard errors can result in oversized tests when there is substantial temporal dependence. This stimulated a second approach based on inconsistent LRV estimators that keep the bandwidth at a fixed fraction of the sample size [cf. Kiefer, Vogelsang, and Bunzel (2000)]. Inference becomes nonstandard and it is shown that fixed-$b$ achieves high-order refinements [e.g., Sun, Phillips, and Jin (2008)] and reduces the oversize problem of HAR tests. However, unlike the classical approach, fixed-$b$ HAR inference is only valid under stationarity.

Recent work by Casini (2021) questioned the performance of HAR inference tests under nonstationarity from a theoretical standpoint. In the past, simulation evidence of serious (e.g., non-monotonic) power or related issues in specific HAR inference contexts were documented by Altissimo and Corradi (2003), Casini (2018), Casini and Perron (2019, 2020a, 2020b), Chan (2020), Chang and Perron (2018), Crainiceanu and Vogelsang (2007), Deng and Perron (2006), Juhl and Xiao (2009), Kim and Perron (2009), Martins and Perron (2016), Perron and Yamamoto (2021) and Vogelsang (1999), among others. Our theoretical results show that these issues occur because the unaccounted nonstationarity alters the spectrum at low frequencies. Each sample autocovariance is upward biased ($d^* > 0$) and the resulting LRV estimators tend to be inflated. When these estimators are used to normalize test statistics, the latter lose power. Interestingly, $d^*$ is independent of $k$ so that the more lags are included the more severe is the problem. Further, by virtue of weak dependence, we have that $\Gamma_T(k) \to 0$ as $k \to \infty$ but $d^* > 0$ across $k$. For these reasons, long bandwidths/fixed-$b$ LRV estimators are expected to suffer most because they use many/all lagged autocovariance. Our theoretical results further show that nonparametric smoothing effectively solves the problem; the DK-HAC estimators from Casini (2021) lead to HAR tests with good size and good power even when existing HAR tests have little or no power.

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This paper makes an independent contribution relative to other recent working papers by two of the authors. Casini and Perron (2021c) establishes minimax MSE bounds for LRV estimation and proposes a new prewhitening procedure robust to nonstationarity. Casini and Perron (2021a) presents change-point tests for time series with an evolutionary spectra that can also be used to eliminate the bias of the local periodogram when there are discontinuities in the spectrum. The paper is organized as follows. Section 2 presents the statistical setting and Section 3 establishes the theoretical results. The implications of our results for HAR inference are analyzed analytically and computationally through simulations in Section 4. Section 5 concludes. The code to implement our method is provided in Matlab, R and Stata languages.

2 Statistical Framework for Nonstationarity

Suppose \( \{V_{t,T}\}_{t=1}^T \) is defined on an abstract probability space \((\Omega, \mathcal{F}, \mathbb{P})\), where \(\Omega\) is the sample space, \(\mathcal{F}\) is the \(\sigma\)-algebra and \(\mathbb{P}\) is a probability measure. In order to analyze time series models that have a time-varying spectrum it is useful to introduce an infill asymptotic setting whereby we rescale the original discrete time horizon \([1, T]\) by dividing each \(t\) by \(T\). Letting \(u = t/T\) and \(T \to \infty\), we define a new time scale \(u \in [0, 1]\) on which as \(T \to \infty\) we observe more and more realizations of \(V_{t,T}\) close to time \(t\). As a notion of nonstationarity, we use the concept of segmented locally stationary processes introduced in Casini (2021). This extends the class of locally stationary processes [cf. Dahlhaus (1997)] that have been used widely in both statistics and economics, often simply referred to as time-varying parameter processes [see e.g., Cai (2007) and Chen and Hong (2012)]. Due to imposed smoothness restrictions, these processes exclude many prominent econometric models that account for time variation in the parameters. For example, structural change and regime switching-type models do not belong to this class because parameter changes occur suddenly at a particular point in time rather than smoothly over short periods. Segmented locally stationary processes allow for a finite number of discontinuities in the spectrum over time. We collect the break dates in the set \(\mathcal{T} \equiv \{T_0^1, \ldots, T_0^m\}\). Let \(i \equiv \sqrt{-1}\). A function \(G(\cdot, \cdot) : [0, 1] \times \mathbb{R} \to \mathbb{C}\) is said to be left-differentiable at \(u_0\) if \(\partial G(u_0, \omega) / \partial u \equiv \lim_{u \to u_0^-} (G(u_0, \omega) - G(u, \omega)) / (u_0 - u)\) exists for any \(\omega \in \mathbb{R}\). Section 2.1 introduces short-memory segmented locally stationary processes. The extension to long memory processes is presented in Section 2.2.

2.1 Short Memory Segmented Locally Stationary Processes

Definition 2.1. A sequence of stochastic processes \(\{V_{t,T}\}_{t=1}^T\) is called segmented locally stationary (SLS) with \(m_0 + 1\) regimes, transfer function \(A^0\) and trend \(\mu\), if there exists a representation

\[
V_{t,T} = \mu_j(t/T) + \int_{-\pi}^{\pi} \exp(i\omega t) A^0_{j,t,T}(\omega) d\xi(\omega), \quad (t = T^0_{j-1} + 1, \ldots, T^0_j),
\]  

(2.1)
for \( j = 1, \ldots, m_0 + 1 \), where by convention \( T_0^0 = 0 \) and \( T_{m_0+1}^0 = T \). The following technical conditions are also assumed to hold:

(i) \( \xi(\lambda) \) is a stochastic process on \([-\pi, \pi]\) with \( \overline{\xi(\omega)} = \xi(-\omega) \) and

\[
\text{cum} \{d\xi(\omega_1), \ldots, d\xi(\omega_r)\} = \varphi \left( \sum_{j=1}^{r} \omega_j \right) g_r(\omega_1, \ldots, \omega_{r-1}) \, d\omega_1 \ldots d\omega_r,
\]

where \( \text{cum} \{ \ldots \} \) denotes the cumulant spectra of \( r \)-th order, \( g_1 = 0 \), \( g_2(\omega) = 1 \), \( |g_r(\omega_1, \ldots, \omega_{r-1})| \leq M_r \) for all \( r \) with \( M_r \) being a constant that may depend on \( r \), and \( \varphi(\omega) = \sum_{j=-\infty}^{\infty} \delta(\omega + 2\pi j) \) is the period \( 2\pi \) extension of the Dirac delta function \( \delta(\cdot) \).

(ii) There exists a constant \( K \) (which depends on \( j \)) and a piecewise continuous function \( A : [0, 1] \times \mathbb{R} \rightarrow \mathbb{C} \) such that, for each \( j = 1, \ldots, m_0 + 1 \), there exists a \( 2\pi \)-periodic function \( A_j : [0, \pi] \rightarrow \mathbb{C} \) with \( A_j(u, -\omega) = \overline{A_j(u, \omega)} \), \( \lambda_j^0 \triangleq \frac{T_j^0}{T} \), and for all \( T \),

\[
A(u, \omega) = A_j(u, \omega) \quad \text{for} \quad \lambda_{j-1}^0 < u \leq \lambda_j^0, 
\]

\[
\sup_{1 \leq j \leq m_0 + 1} \sup_{T_{j-1}^0 < t \leq T_j^0} |A_{j,t,T}(\omega) - A_j(t/T, \omega)| \leq KT^{-1}. \tag{2.3}
\]

(iii) \( \mu_j(t/T) \) is piecewise continuous.

**Assumption 2.1.** (i) \( \{V_{t,T}\} \) is a mean-zero SLS process with \( m_0 + 1 \) regimes; (ii) \( A(u, \omega) \) is twice continuously differentiable in \( u \) at all \( u \neq \lambda_j^0 \), \( j = 1, \ldots, m_0 + 1 \), with uniformly bounded derivatives \( (\partial/\partial u) A(u, \cdot) \) and \( (\partial^2/\partial u^2) A(u, \cdot) \), and Lipschitz continuous in the second component with index \( \vartheta = 1 \); (iii) \( (\partial^2/\partial u^2) A(u, \cdot) \) is Lipschitz continuous at all \( u \neq \lambda_j^0 \) \((j = 1, \ldots, m_0 + 1)\); (iv) \( A(u, \omega) \) is twice left-differentiable in \( u \) at \( u = \lambda_j^0 \) \((j = 1, \ldots, m_0 + 1)\) with uniformly bounded derivatives \( (\partial/\partial u) A(u, \cdot) \) and \( (\partial^2/\partial u^2) A(u, \cdot) \) and has piecewise Lipschitz continuous derivative \( (\partial^2/\partial u^2) A(u, \cdot) \).

We define the time-varying spectral density as \( f_j(u, \omega) \triangleq |A_j(u, \omega)|^2 \) for \( T_{j-1}^0/T < u = t/T \leq T_j^0/T \). Given \( f(u, \omega) \), we can define the local covariance of \( V_{t,T} \) at the rescaled time \( u \) with \( Tu \notin \mathcal{T} \) and lag \( k \in \mathbb{Z} \) as \( c(u, k) \triangleq \int_{-\pi}^{\pi} e^{i\omega k} f(u, \omega) \, d\omega \). The same definition is also used when \( Tu \in \mathcal{T} \) and \( k \geq 0 \). For \( Tu \in \mathcal{T} \) and \( k < 0 \) it is defined as \( c(u, k) \triangleq \int_{-\pi}^{\pi} e^{i\omega k} A(u, \omega) A(u-k/T, -\omega) \, d\omega \).

Next, we impose conditions on the temporal dependence (we omit the second subscript \( T \) when it is clear from the context). Let

\[
\kappa_{V_{t,t}}^{(a_1,a_2,a_3,a_4)}(u, v, w) \triangleq \kappa_{V_{t,t}}^{(a_1,a_2,a_3,a_4)}(t, t + u, t + v, t + w) - \kappa_{V_{t,t}}^{(a_1,a_2,a_3,a_4)}(t, t + u, t + v, t + w)
\]

\[
\triangleq \mathbb{E} \left( V_{t}^{(a_1)} - \mathbb{E}V_{t}^{(a_1)} \right) \left( V_{t+u}^{(a_2)} - \mathbb{E}V_{t+u}^{(a_2)} \right) \left( V_{t+v}^{(a_3)} - \mathbb{E}V_{t+v}^{(a_3)} \right) \left( V_{t+w}^{(a_4)} - \mathbb{E}V_{t+w}^{(a_4)} \right)
\]

\[
- \mathbb{E} \left( V_{t}^{(a_1)} - \mathbb{E}V_{t}^{(a_1)} \right) \left( V_{t+u}^{(a_2)} - \mathbb{E}V_{t+u}^{(a_2)} \right) \left( V_{t+v}^{(a_3)} - \mathbb{E}V_{t+v}^{(a_3)} \right) \left( V_{t+w}^{(a_4)} - \mathbb{E}V_{t+w}^{(a_4)} \right),
\]

\[
\forall \lambda_j^0 < u \leq \lambda_j^0.
\]
where \( \{V_{t,e}\} \) is a Gaussian sequence with the same mean and covariance structure as \( \{V_t\} \), 
\( \kappa_{V_t}^{(a_1,a_2,a_3,a_4)} (u, v, w) \) is the time-4 fourth-order cumulant of \( (V_t^{(a_1)} , V_{t+u}^{(a_2)} , V_{t+v}^{(a_3)} , V_{t+w}^{(a_4)}) \) while \( \kappa_{\tilde{V}_t}^{(a_1,a_2,a_3,a_4)} (t, t+u, t+v, t+w) \) is the time-t centered fourth moment of \( \tilde{V}_t \) if \( \tilde{V}_t \) were Gaussian.

**Assumption 2.2.** (i) \( \sum_{k=-\infty}^{\infty} \sup_{u \in [0,1]} \|c(u,k)\| < \infty \) and \( \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \sup_{u \in [0,1]} |\kappa_{\tilde{V}_t}^{(a_1,a_2,a_3,a_4)} (k, j, l)| < \infty \) for all \( a_1, a_2, a_3, a_4 \leq p \). (ii) For all \( a_1, a_2, a_3, a_4 \leq p \) there exists a function 
\( \tilde{\kappa}_{a_1,a_2,a_3,a_4} : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{R} \) such that \( \sup_{1 \leq j \leq m_0+1} \sup_{0 \leq \lambda_j^0 \leq \lambda_{j+1}^0} |\kappa_{\tilde{V}_t}^{(a_1,a_2,a_3,a_4)} (k, j, l)| \) is twice differentiable in \( u \) at all \( u \neq \lambda_j^0 \) \( j = 1, \ldots, m_0+1 \) with uniformly bounded derivatives \( (\partial/\partial u)^2 \tilde{\kappa}_{a_1,a_2,a_3,a_4} (u, \cdot, \cdot, \cdot) \) and \( (\partial^2/\partial u^2) \tilde{\kappa}_{a_1,a_2,a_3,a_4} (u, \cdot, \cdot, \cdot) \), and twice left-differentiable in \( u \) with uniformly bounded derivatives \( (\partial/\partial u) \tilde{\kappa}_{a_1,a_2,a_3,a_4} (u, \cdot, \cdot, \cdot) \) and \( (\partial^2/\partial u^2) \tilde{\kappa}_{a_1,a_2,a_3,a_4} (u, \cdot, \cdot, \cdot) \), and piecewise Lipschitz continuous derivative \( (\partial^2/\partial u^2) \tilde{\kappa}_{a_1,a_2,a_3,a_4} (u, \cdot, \cdot, \cdot) \).

If \( \{V_t\} \) is stationary then the cumulant condition of Assumption 2.2-(i) reduces to the standard one used in the time series literature [see Andrews (1991)]. Note that \( \alpha \)-mixing and moment conditions imply that the cumulant condition of Assumption 2.2 holds.

### 2.2 Long Memory Segmented Locally Stationary Processes

One of our goals is to show that the sample autocovariances and the periodogram based on short memory SLS processes have properties similar to those of long memory processes. Thus, we need to first define long memory SLS processes and illustrate the properties of these statistics for such processes. Define the backward difference operator \( \Delta V_t = V_t - V_{t-1} \) and \( \Delta^j V_t \) recursively. Long memory features can be expressed as a “pole” in the spectral density at frequency zero. That is, for a stationary process, long memory implies that \( f(\omega) \sim \omega^{-2\vartheta} \) as \( \omega \to 0 \) where \( \vartheta \in (0, 1/2) \) is the long memory parameter. In what follows, \( l \) is some non-negative integer.

**Definition 2.2.** A sequence of stochastic processes \( \{V_{t,T}\} \) is called long memory segmented locally stationary with \( m_0 + 1 \) regimes, transfer function \( A^0 \) and trend \( \mu \), if there exists a representation

\[
\Delta^j V_t = \mu_j (t/T) + \int_{-\pi}^{\pi} \exp(i\omega t) A^0_{j,T}(\omega) d\xi(\omega), \quad \left( t = T_{j-1}^0 + 1, \ldots, T_j^0 \right),
\]

for \( j = 1, \ldots, m_0 + 1 \), where by convention \( T_0^0 = 0 \) and \( T_{m_0+1}^0 = T \), (i) and (iii) of Definition 2.1 hold, and (ii) of Definition 2.1 is replaced by

(ii) There exists two constants \( L_2 > 0 \) and \( D < 1/2 \) (which depend on \( j \)) and a piecewise continuous function \( A : [0, 1] \times \mathbb{R} \to \mathbb{C} \) such that, for each \( j = 1, \ldots, m_0 + 1 \), there exists a \( 2\pi \)-periodic function \( A_j : (\lambda_{j-1}^0, \lambda_j^0] \times \mathbb{R} \to \mathbb{C} \) with \( A_j (u, -\omega) = \overline{A_j (u, \omega)} \),

\[
A(u, \omega) = A_j (u, \omega) \quad \text{for} \quad \lambda_{j-1}^0 < u \leq \lambda_j^0,
\]

where \( \{V_{t,e}\} \) is a Gaussian sequence with the same mean and covariance structure as \( \{V_t\} \), 
\( \kappa_{V_t}^{(a_1,a_2,a_3,a_4)} (u, v, w) \) is the time-4 fourth-order cumulant of \( (V_t^{(a_1)} , V_{t+u}^{(a_2)} , V_{t+v}^{(a_3)} , V_{t+w}^{(a_4)}) \) while \( \kappa_{\tilde{V}_t}^{(a_1,a_2,a_3,a_4)} (t, t+u, t+v, t+w) \) is the time-t centered fourth moment of \( \tilde{V}_t \) if \( \tilde{V}_t \) were Gaussian.
\[
\sup_{1 \leq j \leq m_0 + 1} \sup_{T_{j-1} \leq t \leq T_j} \left| A^0_{j,t,T}(\omega) - A_j(t/T, \omega) \right| \leq L_2 T^{-1} |\omega|^{-D}, \quad (2.6)
\]

and
\[
\sup_{0 \leq u \leq u' \leq T, u \neq \lambda_j^{(j = 1, \ldots, m_0 + 1), \omega}} |A(u, \omega) - A(u', \omega)| \leq L_2 |u - u'| |\omega|^{-D}. \quad (2.7)
\]

The spectral density of \( \{V_{t,T}\} \) is given by
\[
f_j(u, \omega) = |1 - \exp(-i\omega)|^{-2} |A_j(u, \omega)|^{-2}
\]
for \( j = 1, \ldots, m_0 + 1 \). We say that the process \( \{V_{t,T}\} \) has local memory parameter \( \vartheta(u) \in (-\infty, l + 1/2) \) at time \( u \in [0, 1] \) if it satisfies (2.4)-(2.7), and its generalized spectral density \( f_j(u, \omega) \) \((j = 1, \ldots, m_0 + 1)\) satisfies the following condition,
\[
f_j(u, \omega) = |1 - e^{-i\omega}|^{-2\vartheta_j(u)} f^*_j(u, \omega), \quad (2.8)
\]
with \( f^*_j(u, \omega) > 0 \) and
\[
|f^*_j(u, \omega) - f^*_j(u, 0)| \leq L_4 f^*_j(u, \omega) |\omega|^\nu, \quad \omega \in [-\pi, \pi], \quad (2.9)
\]
where \( L_4 > 0 \) and \( \nu \in (0, 2] \).

Definition 2.2 extends Definition 2.1 and Assumption 2.1 by requiring the bound on the smoothness of \( A(\cdot, \omega) \) to depend also on \( |\omega|^{-D} \) thereby allowing a singularity at \( \omega = 0 \). Casini (2021) showed that \( f_j(u, \omega) = |A_j(u, \omega)|^2 \) for \( j = 1, \ldots, m_0 + 1 \). Using similar arguments, we obtain the form \( f_j(u, \omega) \) given in (2.8). See Roueff and von Sachs (2011) for a definition of long memory local stationarity. Definition 2.2 extends their definition to allow for \( m_0 \) discontinuities. We have assumed that breaks in the long memory parameter occur at the same locations as the breaks in the spectrum. This can be relaxed but would provide no added value in this paper.

**Example 2.1.** A time-varying AR fractionally integrated moving average \((p, \vartheta, q)\) process with \( m_0 \) structural breaks satisfies Definition 2.2 with \( \vartheta_j : [0, 1] \rightarrow (-\infty, l + 1/2), \sigma_j : [0, 1] \rightarrow \mathbb{R}_+, \phi_j = [\phi_1, \ldots, \phi_p]' : [0, 1] \rightarrow \mathbb{R}^p \) and \( \theta_j = [\theta_1, \ldots, \theta_q]' : [0, 1] \rightarrow \mathbb{R}^q \) are left-Lipschitz functions for each \( j = 1, \ldots, m_0 + 1 \) such that \( 1 - \sum_{k=1}^p \phi_j \cdot z_k \) does not vanish for all \( u \in [0, 1] \) and \( z \in \mathbb{C} \) such that \( |z| \leq 1 \). Using the latter condition, the local transfer function \( A_j(u; \cdot) \) defines for each \( j \) a causal autoregressive fractionally integrated moving average (ARFIMA\((p, \vartheta(u) - l, q)\) process whose spectral density satisfies the conditions (2.8) and (2.9) with \( \nu = 2 \). Using Lemma 3 in Roueff and von Sachs (2011), condition (2.7) holds with with \( D > \sup_{1 \leq j \leq m_0 + 1} \frac{1}{\lambda_j^{(j - 1, u \leq \lambda_j^0, \omega \vartheta_j(u) - l.}}
\]

Definition 2.2 implies that \( \rho_X(u, k) \triangleq \text{Corr}(X_u, X_{u+k}) \sim C k^{2d_j(u) - 1} \) for \( \lambda_j^{0, u \leq \lambda_j^0, \omega \vartheta_j(u) - l.} \)
and large \( k \) where \( C > 0 \). This means that the rescaled time-\( u \) autocorrelation function (ACF(\( u \))) has a power law decay which implies \( \sum_{k=-\infty}^{\infty} |\rho_X(u, k)| = \infty \) if \( d_j(u) \in (0, 1/2) \).
3 Theoretical Results on Low Frequency Contamination

In this section we establish theoretical results about the low frequency contamination induced by nonstationarity including results covering the case of misspecification and outliers. We first consider the asymptotic properties of two key quantities for inference in time series contexts, i.e., the sample autocovariance and the periodogram. These are defined, respectively, by

\[ \Gamma (k) = \begin{cases} T^{-1} \sum_{t=k+1}^{T} (V_t - \overline{V}) (V_{t-k} - \overline{V}), & k \geq 0, \\ T^{-1} \sum_{t=-k+1}^{-1} (V_{t+k} - \overline{V}) (V_t - \overline{V}), & k < 0, \end{cases} \]  

(3.1)

where \( \overline{V} \) is the sample mean and

\[ I_T (\omega) = \left| \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \exp (-i\omega t) V_t \right|^2, \quad \omega \in [0, \pi], \]

which is evaluated at the Fourier frequencies \( \omega_j = (2\pi j)/T \in [0, \pi] \). In the context of autocorrelated data, hypotheses testing and construction of confidence intervals require estimation of the so-called long-run variance. Traditional HAC estimators are weighted sums of sample autocovariances while frequency domain estimators are weighted sums of the periodograms. Casini (2021) considered an alternative estimate for the sample autocovariance to be used in the DK-HAC estimators,\(^3\) namely,

\[ \hat{\Gamma}_{DK} (k) \triangleq \frac{n_T}{T} \sum_{r=0}^{[T/n_T]} \tilde{c}_T (rn_T/T, k), \]

where \( k \in \mathbb{Z}, n_T \to \infty \) satisfying the conditions given below, and

\[ \tilde{c}_T (rn_T/T, k) = n_{2,T}^{-1} \sum_{s=0}^{n_{2,T}^{-1}} \left( V_{rn_T+k/2-n_{2,T}/2+s+1} - \overline{V}_{rn_T} \right) \left( V_{rn_T+k/2-n_{2,T}/2+s+1-k} - \overline{V}_{rn_T} \right), \]

(3.2)

where \( n_{2,T} \to \infty \) and \( \overline{V}_{rn_T} = n_{2,T}^{-1} \sum_{s=0}^{n_{2,T}^{-1}} V_{rn_T+k/2-n_{2,T}/2+s+1}. \) \( \tilde{c}_T (rn_T/T, k) \) is an estimate of the autocovariance at time \( rn_T \) and lag \( k \), i.e., \( \text{cov} (V_{rn_T}, V_{rn_T-k}) \). One could use a smoothed or tapered version. However, to clearly focus on the main intuition, we prefer to omit these estimates to keep the notation simple. The theoretical results remain the same. The estimate \( \hat{\Gamma}_{DK} (k) \) is an integrated local sample autocovariance. It extends \( \hat{\Gamma} (k) \) to better account for nonstationarity. Similarly, the DK-HAC estimator does not relate to the periodogram but to the local periodogram.

---

\(^3\)The DK-HAC estimators are defined in Section 4.1.
defined by

\[ I_{L,T}(u, \omega) \triangleq \frac{1}{\sqrt{n_T}} \left| \sum_{s=0}^{n_T-1} V_{[Tu]_n} \exp(-i\omega s) \right|^2, \]

where \( I_{L,T}(u, \omega) \) is the (untapered) periodogram over a segment of length \( n_T \) with midpoint \([Tu]\).

We also consider the statistical properties of both \( \hat{\Gamma}_{DK}(k) \) and \( I_{L,T}(u, \omega) \) under nonstationarity. Define \( r_j = (\lambda_j^0 - \lambda_{j-1}^0) \) for \( j = 1, \ldots, m_0 + 1 \) with \( \lambda_0^0 = 0 \) and \( \lambda_{m_0+1}^0 = 1 \). Note that \( \lambda_j^0 = \sum_{s=0}^j r_s \).

### 3.1 The Sample Autocovariance under Nonstationarity

We now establish some asymptotic properties of the sample autocovariance under nonstationarity. We consider the case \( k \geq 0 \) only; the case \( k < 0 \) is similar and omitted. Let

\[ \bar{\mu}_j = r_j^{-1} \int_{\lambda_{j-1}^0}^{\lambda_j^0} \mu_j(u) \, du, \quad \text{for } j = 1, \ldots, m_0 + 1, \]

where \( \mu_j(\cdot) \) is defined in (2.1). We use \( \sum_{j_1 \neq j_2} \) as a shorthand for the double sum \( \sum_{\{j_1, j_2 = 1, \ldots, m_0 + 1, j_1 \neq j_2\}} \).

**Theorem 3.1.** Assume that \( \{V_{t,T}\} \) satisfies Definition 2.1. Under Assumption 2.1-2.2,

\[ \hat{\Gamma}(k) \geq \int_0^1 c(u, k) \, du + d^*_{\text{Sta}} + o_{\text{a.s.}}(1), \quad (3.3) \]

where \( d^* = 2^{-1} \sum_{j_1 \neq j_2} r_{j_1} r_{j_2} (\bar{\mu}_{j_2} - \bar{\mu}_{j_1})^2 \). Further, as \( k \to \infty \), \( \hat{\Gamma}(k) \geq d^* \, \mathbb{P}\text{-a.s.} \) If in addition it holds that \( \mu_j(t/T) = \mu_j \) for \( j = 1, \ldots, m_0 + 1 \), then

\[ \hat{\Gamma}(k) = \int_0^1 c(u, k) \, du + d^*_{\text{Sta}} + o_{\text{a.s.}}(1), \]

where \( d^*_{\text{Sta}} = 2^{-1} \sum_{j_1 \neq j_2} r_{j_1} r_{j_2} (\mu_{j_2} - \mu_{j_1})^2 \) and, as \( k \to \infty \), \( \hat{\Gamma}(k) = d^*_{\text{Sta}} + o_{\text{a.s.}}(1). \)

Theorem 3.1 reveals interesting features. It is easier for illustrative purposes to discuss the case \( \mu_j(t/T) = \mu_j \) for \( j = 1, \ldots, m_0 + 1 \) for which the mean of \( V_t \) in each regime is constant. The theorem states that \( \hat{\Gamma}(k) \) is asymptotically the sum of two terms. The first is the true autocovariance of \( \{V_t\} \) at lag \( k \). The second depends on the difference in the mean across regimes. This term is always positive and it increases in magnitude with the difference in the mean across regimes. Thus, nonstationarity (here in the form of structural breaks in the mean) induces a positive bias. In the next section, we shall discuss cases in which this bias arises as a finite-sample problem and cases where the bias remains even asymptotically. The result that \( \hat{\Gamma}(k) = d^* + o_{\text{a.s.}}(1) \) as \( k \to \infty \) implies that unaccounted nonstationarity generates long memory effects. The intuition
is straightforward. In the previous section we defined a long memory SLS process via the condition \( \sum_{k=-\infty}^{\infty} \Gamma(u, k) \to \infty \) for some \( u \in (0, 1) \). Similarly, a stationary long memory process implies \( \sum_{k=-\infty}^{\infty} \Gamma(k) \to \infty \). The theorem shows that \( \widehat{\Gamma}(k) \) exhibits a similar property. Thus, \( \widehat{\Gamma}(k) \) decays more slowly than for a corresponding short memory stationary process for small lags and then approaches a strictly positive constant \( d^* \) for large lags. A similar result for the case \( \mu_j(t/T) = \mu_j \) was discussed under stationarity in Mikosch and Stărică (2004) to explain long memory in the volatility of financial returns. Their result is driven by abrupt breaks in the second moments of a stationary process. Our result is more general since it allows for general forms of nonstationarity and has many empirical implications.

Theorem 3.1 suggests that any deviation from stationarity can generate a long memory component \( d^* \) or \( d^*_{\text{Sta}} \) that leads to overestimation of the true autocovariance. That is, either a stationary model with breaks or a locally stationary model or a combination of the two can generate these issues. It follows that also the LRV is overestimated. Since the LRV is used to normalized test statistics, this has important consequences for many HAR inference tests that are characterized by deviations from stationarity under the alternative hypotheses. These include tests for forecast evaluation, tests and inference for structural change models, time-varying parameters models and regime switching models. In the context of the linear regression model, \( V_t \) corresponds to the least-squares residuals. Thus, Theorem 3.1 is relevant for regression models with mild forms of misspecification that do not undermine the conditions for consistency of the least-squares estimator. In those cases, the misspecification contaminates the residuals and so generates a long memory component. Examples include exclusion of a relevant regressor uncorrelated with the included regressors, or inclusion of an irrelevant regressor. Also unaccounted nonlinearities and outliers can contaminate the mean of \( V_t \) and therefore contribute to \( d^* \). The difference between \( d^* \) and \( d^*_{\text{Sta}} \) is that the latter is generated only by structural breaks or regime switching.

### 3.2 The Periodogram under Nonstationarity

Classical LRV estimators are weighted averages of periodogram ordinates around the zero frequency. Thus, it is useful to study the behavior of the periodogram as the frequency \( \omega \) approaches zero. We now establish some properties of the asymptotic bias of the periodogram under nonstationarity. We consider the Fourier frequencies \( \omega_l = 2\pi l/T \in (-\pi, \pi) \) for an integer \( l \neq 0 \) (mod \( T \)) and exclude \( \omega_l = 0 \) for mathematical convenience.

**Assumption 3.1.** (i) For each \( j = 1, \ldots, m_0 + 1 \) there exists a \( B_j \in \mathbb{R} \) such that

\[
\left| \sum_{j=1}^{m_0+1} \sum_{t=\lfloor T\lambda_{j-1}^0 + 1}^{\lfloor T\lambda_j^0 \rfloor} \mu_j(t/T) \exp(-i\omega_l t) \right|^2 \geq \sum_{j=1}^{m_0+1} B_j \sum_{t=\lfloor T\lambda_{j-1}^0 + 1}^{\lfloor T\lambda_j^0 \rfloor} \exp(-i\omega_l t), \quad \omega_l \in (-\pi, \pi),
\]
where $B_{j_1} \neq B_{j_2}$ for $j_1 \neq j_2$; (ii) $|\Gamma(u, k)| = C_{u,k}k^{-m}$ for all $u \in [0, 1]$ and all $k \geq C_3 T^\kappa$ for some $C_3 < \infty$, $C_{u,k} < \infty$ (which depends on $u$ and $k$), $0 < \kappa < 1/2$, and $m > 2$.

Part (i) is easily satisfied (e.g., the special case with $\mu_j (t/T) = \mu_j$). Part (ii) is satisfied if $\{V_t\}$ is strong mixing with mixing parameters of size $-2\nu/ (\nu - 1/2)$ for some $\nu > 1$ such that $\sup_{t \geq 1} \mathbb{E} |V_t|^{\nu} < \infty$. This is less stringent than the size condition sufficient for Assumption 2.2-(i).

**Theorem 3.2.** Assume that $\{V_{t,T}\}$ satisfies Definition 2.1. Under Assumption 2.1-2.2 and 3.1,

$$
\mathbb{E} (I_T (\omega_l)) = 2\pi \int_0^1 f(u, \omega_l) du \\
+ \frac{1}{T \omega_l^2} \left[ \sum_{j=1}^{m_0} (B_j - B_{j+1}) \exp \left( -2\pi i l \lambda_j^0 \right) \right]^2 + o(1).
$$

Under Assumption 2.1-2.2 and 3.1-(ii), if $\mu_j (t/T) = \mu_j$ for each $j = 1, \ldots, m_0 + 1$, then

$$
\mathbb{E} (I_T (\omega_l)) = 2\pi \int_0^1 f(u, \omega_l) du \\
+ \frac{1}{T \omega_l^2} \left[ \sum_{j=1}^{m_0} (\mu_j - \mu_{m_0+1}) \exp \left( -2\pi i l \lambda_j^0 \right) \right]^2 + o(1).
$$

In either case, if $T \omega_l^2 \to 0$ as $T \to \infty$ then $\mathbb{E} (I_T (\omega_l)) \to \infty$ for many values in $\{\omega_l\}$ as $\omega_l \to 0$.

The theorem suggests that for small frequencies $\omega_l$ close to 0, the periodogram attains very large values. This follows because the first term of (3.4) is bounded for all frequencies $\omega_j$. Since $B_1, \ldots, B_{m_0+1}$ are arbitrary, the order of the second term of (3.4) is $O((T \omega_l^2)^{-1})$. Note that as $\omega_l \to 0$ there are some values $l$ for which the corresponding term involving $|\cdot|^2$ on the right-hand side of (3.4) is equal to zero. In such cases, $\mathbb{E} (I_T (\omega_l)) \geq 2\pi \int_0^1 f(u, \omega_l) du > 0$. For other values of $\{l\}$ as $\omega_l \to 0$, the second term of (3.4) diverges to infinity. Thus, considering the behavior of the sequence $\{\mathbb{E} (I_T (\omega_l))\}$ as $\omega_l \to 0$, it generally takes arbitrary unbounded values except for some $\omega_l$ for which $\mathbb{E} (I_T (\omega_l))$ is bounded below by $2\pi \int_0^1 f(u, \omega_l) du > 0$. This behavior is consistent with long memory as discussed in the previous section. A SLS process with long memory has an unbounded local spectral density $f(u, \omega)$ as $\omega \to 0$ for some $u \in [0, 1]$. Since $f(\cdot, \cdot)$ cannot be negative, it follows that $\int_0^1 f(u, \omega) du$ is also unbounded as $\omega \to 0$. Theorem 3.2 suggests that nonstationarity consisting of time-varying first moment results in a periodogram sharing features of the periodogram of a long memory series around the zero frequency. Since the periodogram behavior around the zero frequency characterizes a long memory process, nonstationarity can generate long memory effects.
3.3 The Sample Local Autocovariance under Nonstationarity

We now consider the behavior of \( \hat{c}_T (rn_T/T, k) \) as defined in (3.2) for fixed \( k \) as well as for \( k \to \infty \).

**Theorem 3.3.** Assume that \( \{V_{t,T}\} \) satisfies Definition 2.1, \( n_T, n_{2,T} \to \infty \) with \( n_T/T \to 0 \) and \( n_T/n_{2,T} \to 0 \). Under Assumption 2.1-2.2,

(i) for any \( u \in (0, 1) \) such that \( T_j^0 \notin [\lfloor Tu \rfloor + k/2 - n_{2,T}/2 + 1, \ldots, \lceil Tu \rceil + k/2 + n_{2,T}/2] \) for all \( j = 1, \ldots, m_0 \), \( \hat{c}_T (u, k) = c(u, k) + o_p (1) \);

(ii) for any \( u \in (0, 1) \) such that \( T_j^0 \in [\lfloor Tu \rfloor + k/2 - n_{2,T}/2 + 1, \ldots, \lceil Tu \rceil + k/2 + n_{2,T}/2 + 1] \) for some \( j = 1, \ldots, m_0 \) we have two sub-cases: (a) if \( (T_j^0 - (\lfloor Tu \rfloor + k/2 - n_{2,T}/2 + 1)))/n_{2,T} \to \gamma \) or \( ((\lfloor Tu \rfloor + k/2 + n_{2,T}/2 + 1) - T_j^0)/n_{2,T} \to \gamma \) with \( \gamma \in (0, 1) \), then

\[
\hat{c}_T (u, k) \geq \gamma c (\lambda_j^0, k) + (1 - \gamma) c(u, k) + \gamma (1 - \gamma) \left( \mu_j (\lambda_j^0) - \mu_{j+1} (u) \right)^2 + o_p (1).
\]

(b) if \( (T_j^0 - (T_u + k/2 - n_{2,T}/2 + 1))/n_{2,T} \to 0 \) or \( ((\lfloor Tu \rfloor + k/2 + n_{2,T}/2 + 1) - T_j^0)/n_{2,T} \to 0 \), then \( \hat{c}_T (u, k) = c(u, k) + o(1) \).

Further, if there exists an \( r = 1, \ldots, \lceil T/n_T \rceil \) such that there exists a \( j = 1, \ldots, m_0 \) with \( T_j^0 \in [rn_T + k/2 - n_{2,T}/2 + 1, \ldots, rn_T + k/2 + n_{2,T}/2 + 1] \) satisfying (ii-a), then as \( k \to \infty \), \( \Gamma_{DK} (k) \geq d_T P \cdot a.s. \) where \( d_T = (n_T/T) \gamma (1 - \gamma) (\mu_j (\lambda_j) - \mu_{j+1} (u))^2 > 0 \) and \( d_T \to 0 \) as \( T \to \infty \).

The theorem shows that the behavior of \( \hat{c}_T (u, k) \) depends on whether a change-point in mean is present or not, and if present whether it is close enough to \( \lfloor Tu \rfloor \) or not. For a given \( u \in (0, 1) \) and \( k \in \mathbb{Z} \), if the condition of part (i) of the theorem holds, then \( \hat{c}_T (u, k) \) is consistent for \( \text{cov}(V_{[Tu]} V_{[Tu] - k}) = c(u, k) + O (T^{-1}) \) [see Casini (2021)]. If a change-point falls close to the either boundary of the window \( [\lfloor Tu \rfloor + k/2 - n_{2,T}/2 + 1, \ldots, \lceil Tu \rceil + k/2 + n_{2,T}/2 + 1] \), as specified in case (ii-b) then \( \hat{c}_T (u, k) \) remains consistent. The only case in which a non-negligible bias arises is when the change-point falls in the neighborhood around \( \lfloor Tu \rfloor \) sufficiently far from either boundary. This represents case (ii-a), for which a biased estimate results. However, the bias vanishes asymptotically. Since \( \Gamma_{DK} (k) \) is an average of \( \hat{c}_T (rn_T, k) \) over blocks \( r = 1, \ldots, \lceil T/n_T \rceil \), if case (ii-a) holds then \( \Gamma_{DK} (k) \geq d_T^* \) as \( k \to \infty \) but \( d_T^* \to 0 \) as \( T \to \infty \). Thus, comparing this result with Theorem 3.1, in practice the long memory effects are unlikely to occur when using \( \hat{c}_T (u, k) \) instead of \( \hat{c}_T (u, k) \). Furthermore, one can avoid altogether this issue for \( \hat{c}_T (rn_T, k) \) by appropriately choosing the blocks \( r = 1, \ldots, \lceil T/n_T \rceil \). A procedure was proposed in Casini (2021) using the methods developed in Casini and Perron (2021a). Another way to see that \( \Gamma_{DK} (k) \) suffers less from these problems is to look at the form of \( \hat{c}_T (rn_T, k) \). Usually one would use a kernel or a taper to assign more weight to observations that are close to \( \lfloor Tu \rfloor \). This automatically would reduce the contamination which arises from mixing observations belonging to two different regimes because the shorter regime would be down-weighted by the kernel or taper.
3.4 The Local Periodogram under Nonstationarity

We now study the asymptotic properties of $I_{L,T}(u, \omega)$ as $\omega \to 0$ for $u \in [0, 1]$. We consider the Fourier frequencies $\omega_l = 2\pi l/n_T \in (-\pi, \pi)$ for an integer $l \neq 0 \text{ (mod } n_T)$. We need the following high-level conditions.

Assumption 3.2. (i) For each $u \in [0, 1]$ with $T_j^0 \in \lfloor [Tu] - n_T/2 + 1, \ldots, [Tu] + n_T/2 \rfloor$ there exist $B_j \in \mathbb{R}$, $j = 1, \ldots, m_0$ with $B_{j1} \neq B_{j2}$ for $j_1 \neq j_2$ such that

$$\left| \sum_{s=0}^{n_T-1} \mu((\lfloor Tu \rfloor - n_T/2 + s + 1)/T) \exp(-i\omega_l s) \right|^2 \geq \left| B_j \sum_{s=0}^{n_T-1} \exp(-i\omega_l s) + B_{j+1} \sum_{s=T_j^0 - (\lfloor Tu \rfloor - n_T/2)}^{n_T-1} \exp(-i\omega_l s) \right|^2.$$

(ii) $\sup_{u \in [0,1]} (\partial^2/\partial u^2) f(u, \omega)$ is continuous in $\omega$.

Theorem 3.4. Assume that $\{V_{t,T}\}$ satisfies Definition 2.1 and that $n_T \to \infty$ with $n_T/T \to 0$. Under Assumption 2.1-2.2, 3.1-(ii) and 3.2,

(i) for any $u \in (0, 1)$ such that $T_j^0 \notin \lfloor [Tu] - n_T/2 + 1, \ldots, [Tu] + n_T/2 \rfloor$ for all $j = 1, \ldots, m_0$, $\mathbb{E}(I_{L,T}(u, \omega_l)) \geq f(u, \omega_l)$ as $\omega_l \to 0$;

(ii) for any $u \in (0, 1)$ such that $T_j^0 \in \lfloor [Tu] - n_T/2 + 1, \ldots, [Tu] + n_T/2 \rfloor$ for some $j = 1, \ldots, m_0$ we have two sub-cases: (a) if $(T_j^0 - (\lfloor Tu \rfloor - n_T/2 + 1))/n_T \to \gamma$ or $(T_j^0 - (\lfloor Tu \rfloor + n_T/2 + 1))/n_T \to \gamma$ with $\gamma \in (0, 1)$, and $n_T\omega_l^2 \to 0$ as $T \to \infty$, then $\mathbb{E}(I_{L,T}(u, \omega)) \to \infty$ for many values in the sequence $\{\omega_l\}$ as $\omega_l \to 0$; (b) if $(T_j^0 - (\lfloor Tu \rfloor - n_T/2 + 1))/n_T \to 0$ or $(T_j^0 - (\lfloor Tu \rfloor + n_T/2 + 1))/n_T \to 0$, then $\mathbb{E}(I_{L,T}(u, \omega_l)) \geq f(u, \omega_l)$ as $\omega_l \to 0$.

It is useful to compare Theorem 3.4 with Theorem 3.2. Unlike the periodogram, the asymptotic behavior of the local periodogram as $\omega_l \to 0$ depends on the vicinity of $u$ to $\lambda_j^0$ ($j = 1, \ldots, m_0$). Since $I_{L,T}(u, \omega_l)$ uses observations in the window $[Tu - n_T/2 + 1, \ldots, Tu + n_T/2]$, if no discontinuity in the mean occurs in this window then $I_{L,T}(u, \omega_l)$ is asymptotically unbiased for the spectral density $f(u, \omega_l)$ and therefore bounded below by it. More complex is its behavior if some $T_j^0$ falls in the window $[Tu - n_T/2 + 1, \ldots, Tu + n_T/2]$. The theorem shows that if $T_j^0$ is close to the boundary, as indicated in case (ii-b), then $I_{L,T}(u, \omega_l)$ is bounded below by $f(u, \omega_l)$, similarly to case (i). If instead $T_j^0$ falls sufficiently close to the mid-point $Tu$, as indicated in case (ii-a), then $\mathbb{E}(I_{L,T}(u, \omega)) \to \infty$ for many values in the sequence $\{\omega_l\}$ as $\omega_l \to 0$ provided it satisfies $n_T\omega_l^2 \to 0$ as $T \to \infty$. Hence, unless $T\lambda_j^0$ is close to $Tu$, the local periodogram $I_{L,T}(u, \omega_l)$ behaves very differently from the periodogram $I_T(\omega_l)$. Accordingly, nonstationarity is unlikely to generate long memory effects if one uses the local periodogram. Further, if one uses preliminary inference procedures for the detection and estimation of the discontinuities in the spectrum and for the
estimation of their locations, then one can construct the window efficiently and avoid \( T_0 \) being too close to \( T_u \). Such procedures have been proposed recently in Casini and Perron (2021a).

## 4 Consequences for HAR Inference

In this section we discuss the implications of the theoretical results from Section 3 for inference in the context of potentially autocorrelated data (i.e., HAR inference). We separate the discussion into two parts. We first discuss HAR inference tests for which the issues of low frequency contamination arise as a finite-sample problem. Then we discuss HAR inference tests for which the results presented in the previous section apply even asymptotically. We begin with a review of HAR inference methods and their connection to the estimates considered above.

### 4.1 HAR Inference Methods

There are two main approaches for HAR inference which differ on whether the long-run variance estimator is consistent or not. The classical approach relies on consistency which results in HAC standard errors [cf. Newey and West (1987; 1994) and Andrews (1991)]. Classical HAC standard errors require estimation of the long-run variance defined as \( J \triangleq \lim_{T \to \infty} J_T \) where 
\[
J_T = T^{-1} \sum_{s=1}^{T} \sum_{t=1}^{T} \mathbb{E}(V_s V_t').
\]

The form of \( \{V_t\} \) depends on the specific problem under study. For example, for a \( t \)-test on a regression coefficient in the linear model 
\[
y_t = x_t' \beta_0 + e_t \quad (t = 1, \ldots, T)
\]
we have \( V_t = x_t e_t \). Classical HAC estimators take the following form,
\[
\hat{J}_{\text{Cla},T} \triangleq T^{-1} \sum_{k=-T+1}^{T-1} K_1(b_{1,T} k) \hat{\Gamma}(k),
\]

where \( \hat{\Gamma}(k) \) is given in (3.1) with \( \hat{V}_t = x_t \hat{e}_t \) where \( \{\hat{e}_t\} \) are the least-squares residuals, \( K_1(\cdot) \) is a real-valued kernel and \( b_{1,T} \) is bandwidth parameter. One can use the the Bartlett kernel, advocated by Newey and West (1987), or the quadratic spectral kernel as suggested by Andrews (1991), or any other kernel suggested in the literature, see e.g. Ng and Perron (1996). Under \( b_{1,T} \to 0 \) at an appropriate rate, we have \( \hat{J}_{\text{Cla},T} \xrightarrow{p} J \). Hence, equipped with \( \hat{J}_{\text{Cla},T} \), HAR inference is standard because HAR tests follow asymptotically standard distributions. This is the simplest approach.

It was shown that classical HAC standard errors can result in oversized tests when there is substantial temporal dependence [e.g., Andrews (1991)]. This stimulated a second approach based on inconsistent long-run variance estimators that keep the bandwidth at some fixed fraction of the sample size [cf. Kiefer, Vogelsang, and Bunzel (2000)], e.g., using all autocovariances, so that 
\[
\hat{J}_{\text{KVB},T} \triangleq T^{-1} \sum_{t=1}^{T} \sum_{s=1}^{T} (1 - |t-s|/T) V_t V_s
\]
which is equivalent to the Newey-West estimator with \( b_{1,T} = T^{-1} \), in which case \( \hat{J}_{\text{KVB},T} \) is an inconsistent estimate of \( J \). \( \hat{J}_{\text{KVB},T} \) is essentially a weighted average of the periodogram ordinates with weights that do not spread out as \( T \to \infty \).
Because of the inconsistency, inference is nonstandard and HAR tests do not have asymptotically standard distributions. The validity of fixed-\(b\) HAR inference rests on stationarity. Many authors have considered modifications of \(J_{\text{KVB},T}\). However, the one that leads to HAR inference tests that are least oversized is the original \(J_{\text{KVB},T}\) [see Casini and Perron (2021c)]. Here for comparison we also report the equally weighted cosine (EWC) estimator of Lazarus, Lewis, and Stock (2020). This is an orthogonal series estimators that use large-bandwidths,

\[
\widehat{J}_{\text{EWC},T} \triangleq B^{-1} \sum_{j=1}^{B} \Lambda_j^2, \quad \text{where} \quad \Lambda_j = \sqrt{\frac{2}{T}} \sum_{t=1}^{T} V_t \cos \left( \pi j \left( \frac{t-1/2}{T} \right) \right)
\]

with \(B\) some fixed integer. Assuming \(B\) satisfies some conditions, under a fixed-\(b\) asymptotic framework a \(t\)-test normalized by \(\widehat{J}_{\text{EWC},T}\) follows a \(t_B\) distribution.

Recently, a new HAC estimator was proposed in Casini (2021). Motivated by the signs of low frequency contamination of existing long-run variance estimators, he proposed a double kernel HAC (DK-HAC) estimator. This is defined as

\[
\widehat{J}_{\text{DK},T} \triangleq \sum_{k=-T+1}^{T-1} K_1 (b_{1,T} k) \hat{\Gamma}_{\text{DK}} (k),
\]

where \(b_{1,T}\) is a bandwidth sequence and \(\hat{\Gamma}_{\text{DK}} (k)\) defined in Section 3 with \(\hat{c}_T (\cdot, k)\) replaced by

\[
\hat{c}_{\text{DK},T} \left( r n_T / T, k \right) = \begin{cases} 
(T b_{2,T})^{-1} \sum_{s=k+1}^{T} K_2 \left( \frac{((r+1)n_T - (s-k)/2)}{b_{2,T}} \right) V_s V_{s-k}, & k \geq 0 \\
(T b_{2,T})^{-1} \sum_{s=-k-1}^{T} K_2 \left( \frac{((r+1)n_T -(s+k)/2)}{b_{2,T}} \right) V_{s+k} V_s, & k < 0,
\end{cases}
\]

with \(K_2\) a real-valued kernel and \(b_{2,T}\) a bandwidth sequence. Note that \(\hat{c}_{\text{DK},T}\) and \(\hat{c}_T\) are asymptotically equivalent and the results of Section 3 continue to hold for \(\hat{c}_{\text{DK},T}\). More precisely, \(\hat{c}_T\) is a special case of \(\hat{c}_{\text{DK},T}\) with \(K_2\) being a rectangular kernel and \(n_{2,T} = T b_{2,T}\). This approach falls in the first category of standard inference \(J_{\text{DK},T} \overset{p}{\to} J\) and HAR tests normalized by \(\widehat{J}_{\text{DK},T}\) follow standard distribution asymptotically. Additionally, Casini and Perron (2021c) proposed prewhitened DK-HAC (\(\hat{J}_{\text{pw,DK},T}\)) estimator that improves the size control of HAR tests normalized by \(\widehat{J}_{\text{DK},T}\). The estimator \(\hat{J}_{\text{pw,DK},T}\) applies a prewhitening transformation to the data before constructing \(\widehat{J}_{\text{DK},T}\) and enjoys the same asymptotic properties as the non-prewhitened DK-HAC estimator \(\widehat{J}_{\text{DK},T}\). Due to their ability to more flexibly account for nonstationarity, Casini (2021) and Casini and Perron (2021c) demonstrate that \(\hat{J}_{\text{DK},T}\) and \(\hat{J}_{\text{pw,DK},T}\) have superior power properties relative to the other estimators mentioned above. In terms of size, \(\hat{J}_{\text{pw,DK},T}\) performs better than \(\hat{J}_{\text{Cla},T}\) and \(\hat{J}_{\text{DK},T}\), and is competitive with \(\hat{J}_{\text{KVB},T}\) when the latter works well.\(^4\) We include \(\hat{J}_{\text{DK},T}\)

\[^4\text{There some empirical cases where } \hat{J}_{\text{KVB},T} \text{ does not work well in terms of both size and power. In those cases our method works well. For the cases where } \hat{J}_{\text{KVB},T} \text{ leads to tests that have good size, our method is competitive.}\]
and $\hat{J}_{pw,DK,T}$ in our simulations below. We report the results only for the DK-HAC estimators that do not use the pre-test for discontinuities in the spectrum [cf. Casini and Perron (2021a)] because we do not want the results to be affected by the pre-test. Since the pre-test improves the results, what we report here are worst-case results for the DK-HAC estimators.

### 4.2 Small-Sample Low Frequency Contamination

We now discuss situations in which the low frequency contamination arises as a small-sample problem. These comprises situations where $d^* \approx 0$ asymptotically but a consistent estimate of $d^*$ satisfies $\hat{d}^* > 0$ in finite-sample. In these situations, no bias due to long memory effects should occur asymptotically but can have an effect with a small sample-size. We begin with a simple model which involves a zero-mean SLS process with changes in persistence. The combination of nonstationarity and serial dependence generate long memory effects because $\hat{d}^* > 0$.

We specify $\{V_t\}$ as a SLS process given by a two-regimes zero-mean time-varying AR(1), labeled model M1. That is, $V_t = 0.9V_{t-1} + u_t, \ u_t \sim i.i.d. \mathcal{N}(0, 1)$ for $t = 1, \ldots, T^0_1$ with $T^0_1 = T\lambda^0_1$, and $V_t = \rho(t/T) V_{t-1} + u_t, \ \rho(t/T) = 0.3 (\cos (1.5 - \cos (t/T)))$, $u_t \sim i.i.d. \mathcal{N}(0, 0.5)$ for $t = T^0_1 + 1, \ldots, T$. Note that $\rho(\cdot)$ varies between 0.171 and 0.263. We set $\lambda^0_1 = 0.1$ and $T = 200$. A plot of $\{V_t\}$ is reported in Figure 1. Note that $\mathbb{E}(V_t) = 0$ for all $t$ and so $d^* = d^*_{\text{sta}} = 0$. However, if we replace $\pi_1$ and $\pi_2$ by $\nu_1$ and $\nu_2$, respectively, where $\nu_1 = (T^0_1)^{-1} \sum^T_{t=1} V_t = 1.27$ and $\nu_2 = (T - T^0_1)^{-1} \sum^{T}_{t=T^0_1+1} V_t = -0.03$, then our estimate $\hat{d}^*$ of $d^*$ would be different from zero and can generate a finite-sample bias which can give rise to effects akin to long memory.

We first look at the behavior of the sample autocovariance $\hat{\Gamma}(k)$. We compare it with the theoretical autocovariance $\Gamma_T(k) = T^{-1} \sum^T_{t=k+1} \mathbb{E}(V_t V_{t-k})$. The latter is equal to $\Gamma_T(k) \approx \lambda^0_1 0.9^k / (1 - 0.81) + \int_{\lambda^0_1}^{1} c(u, k) du$. We can compute $\Gamma_T(k)$ numerically using,

$$c(u, k) = \int_{-\pi}^{\pi} e^{iku} f(u, \omega) d\omega$$

$$= \int_{-\pi}^{\pi} e^{iku} \frac{0.5}{2\pi} (1 + \rho(u)^2 - 2\rho(u) \cos(\omega))^{-1} d\omega.$$

Table 1 reports $\Gamma_T(k), \hat{\Gamma}(k), \hat{\Gamma}(k) - \hat{d}^*$ and $\hat{\Gamma}_{DK}(k)$ for several values of $k$. It is known that the autocovariance estimates are quite noisy in general. However, it is still possible to discern some patterns. For all $k$, $\hat{\Gamma}(k)$ largely overestimates $\Gamma_T(k)$. This is consistent with Theorem 3.1 which suggests that this is due to the bias $d^* > 0$. This is also supported by the bias-corrected estimate $\hat{\Gamma}(k) - \hat{d}^*$ which is accurate in approximating $\Gamma_T(k)$. This is especially so for small $k$. In general, Theorem 3.1 provides excellent approximations confirming that $\hat{\Gamma}(k)$ suffers from low frequency contaminations. Theorem 3.3 suggests that this issue should not occur for $\hat{\Gamma}_{DK}(k)$. In fact, $\hat{\Gamma}_{DK}(k)$ is more accurate than $\hat{\Gamma}(k)$ (except for $k = 0$). For $k \geq 20$ the form of $\Gamma_T(k)$ is different, because
$T^0_1 = 20$, and is simply given by the autocovariance of $V_t$ for $t \geq 21$ (i.e., the second regime). Thus, $\Gamma_T (k) \approx 0$ for $k \geq 20$ whereas $\hat{\Gamma} (k)$ is often small but positive. In contrast, $\hat{\Gamma}_{DK} (k) \approx 0$ for $k \geq 20$ thereby confirming that $\hat{\Gamma}_{DK} (k)$ does not suffer from low frequency contamination. These results are confirmed in Figure 2 which plots the autocorrelation function (ACF) of $V_{1,t} = V_t$ ($t = 1, \ldots, 20$), $V_{2,t} = V_t$ ($t = 21, \ldots, 200$) and $V_t$ ($t = 1, \ldots, 200$). Although the ACF of $V_t$ should be a weighted average of the ACF of $V_{1,t}$ and of $V_{2,t}$, the ACF of $V_t$ in the bottom panel shows much higher persistence than either ACF in the top ($V_{1,t}$) and mid ($V_{2,t}$) panels. This is odd since $V_{1,t}$ is a highly persistent series. Further, it shows that the dependence is essentially always positive. This is also odd. These features are consistent with our theory which suggests that nonstationarity makes $V_t$ appear more persistent and that the bias is positive. Other examples involve $V_t$ given by the least-squares regression residuals under mild forms of misspecification that do not undermine the conditions for consistency of the least-squares estimator. For example, exclusion of a relevant regressor uncorrelated with the included regressors, or inclusion of an irrelevant regressor. Another example involves $V_t$ obtained after applying some de-trending techniques where the fitted model is not correctly specified (e.g., the data follow a nonlinear trend but one removes a linear trend). A final example is the case of outliers because outliers influences the mean of $V_t$ and therefore $d^*$. What is especially relevant is whether this evidence of long memory feature has any consequence for HAR inference. We obtain the empirical size and power for a $t$-test on the intercept normalized by several LRV estimators for the model $y_t = \delta + V_t$ with $\delta = 0$ under the null and $\delta > 0$ under the alternative hypothesis. In addition to model M1, we consider other models: M2 involves a locally stationary $V_t = \rho (t/T) V_{t-1} + u_t$, $\rho (t/T) = 0.7(\cos (1.5t/T))$, $u_t \sim$ i.i.d. $\mathcal{N} (0, 0.5)$; M3 is the same as M2 with outliers $V_t \sim \text{Uniform} (\zeta, 10\zeta)$ for $t = T/4, T/2, 3T/4$ where $\zeta = -1/(\sqrt{2} \text{erfc}^{-1} (3/2)) \text{med} (|V - \text{med} (V)|)$ where $\text{erfc}^{-1}$ is the inverse complementary error function, $\text{med} (\cdot)$ is the median and $V = (V_t)_{t=1}^T$; M4 involves a locally stationary model with periods of strong persistence where $V_t = \rho (t/T) V_{t-1} + u_t$, $\rho (t/T) = 0.95(\cos (1.5t/T))$, $u_t \sim$ i.i.d. $\mathcal{N} (0, 0.4)$. Note that $\rho (\cdot)$ varies between 0.7 and 0.044 in M3 and between 0.95 and 0.06 in M4.

We consider the DK-HAC estimators with and without prewhitening ($\hat{J}_{DK,T}$, $\hat{J}_{DK,\text{pw},SLS,T}$, $\hat{J}_{DK,\text{pw},SLS,F,T}$) of Casini (2021) and Casini and Perron (2021c), respectively; Andrews’s (1991) HAC estimator with and without the prewhitening procedure of Andrews and Monahan (1992); Newey and West’s (1987) HAC estimator with the usual rule to select the number of lags (i.e., $b_{1,T} = 1/(0.75T^{1/3})$; Newey-West with the fixed-$b$ method of Kiefer, Vogelsang, and Bunzel (2000) (labeled KVB); and the Empirical Weighted Cosine (EWC) of Lazarus, Lewis, Stock, and Watson (2018). For the DK-HAC estimators we use the data-dependent methods for the bandwidths, kernels and choice of $n_T$ as proposed in Casini (2021) and Casini and Perron (2021c), which are

\footnote{We follow the literature on outlier detection for continuous functions and use the median absolute deviation to generate the outlier. This notion used in this literature does not deem a value smaller than $\zeta$ as an outlier.}
optimal under mean-squared error (MSE).\(^6\) Hence, we set \( \hat{b}_{1,T} = 0.6828(\hat{\phi} (2) T \hat{b}_{2,T})^{-1/5} \) where

\[
\hat{\phi} (2) = \sum_{r=1}^{p} p^{-1} \left( 18 \left( \frac{n_T}{T} \right)^{\frac{T/n_{3,T}}{T}} \sum_{j=0}^{\left\lfloor T/n_{3,T} \right\rfloor - 1} \left( \frac{(jn_T + 1) / T}{(1 - \hat{a}_1^{(r)}) ((jn_T + 1) / T)^4} \right)^2 \right) / \sum_{j=0}^{\left\lfloor T/n_{3,T} \right\rfloor - 1} \left( (jn_T + 1) / T \right)^2,
\]

with

\[
\hat{a}_1^{(r)} (u) = \sum_{j=t-n_{2,T}+1}^{t} \left( \frac{\hat{V}_j^{(r)} \hat{V}_{j-1}^{(r)}}{\sum_{j=t-n_{2,T}+1}^{t} \left( \hat{V}_{j-1}^{(r)} \right)^2} \right)^{1/2},
\]

\[
\hat{\sigma}^{(r)} (u) = \left( \sum_{j=t-n_{2,T}+1}^{t} \left( \hat{V}_j^{(r)} - \hat{a}_1^{(r)} (u) \hat{V}_{j-1}^{(r)} \right)^2 \right)^{1/2}.
\]

and \( \hat{b}_{2,T} = (n_T / T) \sum_{j=1}^{T/n_{T}} \hat{b}_{2,T} (rn_T / T), \hat{b}_{2,T} (u) = 1.6786(\hat{D}_1 (u))^{-1/5}(\hat{D}_2 (u))^{1/5}T^{-1/5} \) where \( \hat{D}_2 (u) \triangleq p^{-1} \sum_{r=1}^{p} \sum_{\left\lfloor T/n_{4/25} \right\rfloor} \mathbb{E}_{\text{DK},T} (u, l) [\mathbb{E}_{\text{DK},T}^{(r)} (u, l)] + \sum_{\left\lfloor T/n_{4/25} \right\rfloor} \mathbb{E}_{\text{DK},T} (u, l, 2k) \) and

\[
\hat{D}_1 (u) \triangleq (\left\lfloor S_\omega \right\rfloor)^{-1} \sum_{s \in S_\omega} \left\lfloor \frac{3}{\pi} (1 + 0.8(\cos 1.5 \cos 4\pi u) \exp(-i\omega_s))^{-4} (0.8(-4\pi \sin(4\pi u))) \exp(-i\omega_s) \right\rfloor \right\rfloor ^3 (0.8(-16\pi^2 \cos(4\pi u))) \exp(-i\omega_s) ^2,
\]

with \( \left\lfloor S_\omega \right\rfloor \) being the cardinality of \( S_\omega \) and \( \omega_{s+1} > \omega_s, \omega_1 = -\pi, \omega_{|S_\omega|} = \pi \). We set \( n_T = T^{0.6}, S_\omega = \{-\pi, -3, -2, -1, 0, 1, 2, 3, \pi\} \). \( K_1 (\cdot) \) is the QS kernel and \( K_2 (x) = 6x (1 - x) \) for \( x \in [0, 1] \).

Table 2-5 report the results. The t-test normalized by Newey and West’s and Andrews’s (1991) prewhitened HAC estimators are excessively oversized.\(^7\) Andrews’s (1991) HAC estimator is slightly undersized while KVB’s fixed-b and EWC are severely undersized.\(^8\) These outcomes arise from nonstationarity and is consistent with our theoretical analysis of \( \hat{\Gamma} (k) \). Since \( \hat{\Gamma} (k) > 0 \) for many large lags \( k \), the KVB’s fixed-b and EWC’s estimators that include many lags (i.e., long bandwidths) are inflated and reduce the magnitude of the test statistic even under the null

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\(^6\)See Belotti et al. (2021) for an alternative data-dependent method and for some comparisons.

\(^7\)This is not in contradiction with our theoretical results. The prewhitening of Andrews and Monahan (1992) is unstable when there is nonstationarity as shown by Casini and Perron (2021c). The reason is that there is a bias both in the whitening and the recoloring stages. The biases have opposite signs so that here the underestimation of the LRV dominates. Newey-West uses a fixed rule for determining the number of lags. The number of included lags is small. This estimator is known to be largely oversized when the data are stationary with high dependence. Our results say that the included sample autocovariances may be inflated if there is nonstationarity. However, given that the fixed rule selects a small number of lags then nonstationarity results in a smaller oversize problem.

\(^8\)In general, Andrews’s (1991) HAC estimator leads to tests that are oversized when the data are stationary with strong dependence. Here nonstationarity reduces the oversize problem. It follows a similar argument as for the Newey-West estimator even though Andrews’s (1991) HAC estimator uses a data-dependent method for the selection of the number of lags. Thus, it selects more lags than the ones suggested by the fixed rule. Consequently, more sample autocovariances are overestimated and this helps to reduce its oversize problem.
hypotheses. For the $t$-test on the intercept, $\hat{J}_{DK,T}$ can be oversized when there is strong dependence, as shown in Table 2. However, the prewhitened DK-HAC estimators $\hat{J}_{DK,pw,SLS,T}$ and $\hat{J}_{DK,pw,SLS,\mu,T}$ show very accurate rejection rates. Overall, inspection of the size properties suggests that the DK-HAC estimators do not suffer from low frequency contamination which, in contrast, affects the LRV estimators that rely on the full sample estimates $\hat{\Gamma}(k)$ or equivalently on $I_T(\omega)$ (e.g., the EWC). The same issue also affects the power properties of the tests. The KVB’s fixed-$b$ and EWC’s estimators suffer from relatively large power losses. The power of Newey-West’s and Andrews’s (1991) prewhitened HAC is not comparable because they are significantly oversized. The DK-HAC estimators have the best power, the second best being Andrews’s (1991) HAC estimator.

Turning to M2, Table 3 shows even larger size distortions and power losses for KVB’s fixed-$b$ and EWC’s estimators. All the DK-HAC estimators display accurate size control and good power. Newey and West’s and Andrews’ (1991) prewhitened HAC estimators are again excessively oversized. Andrews’s (1991) HAC estimator sacrifices some power relative to the DK-HAC even though the margin is not high. For model M3-M4, Table 4-5 show that Andrews’s (1991) HAC estimator also suffers from strong size distortions and power losses, thus sharing the same issues as KVB’s fixed-$b$ and EWC’s estimators. In model M4, even Newey and West’s and Andrews’s (1991) prewhitened HAC estimators are undersized and have relatively low power. The DK-HAC estimators perform best both in terms of size and power. Table 2-5 suggest that the low frequency contamination can equally arise from different forms of nonstationarity. Overall, the evidence for quite substantial underejection and power losses in model M1-M4 for the existing LRV estimators is consistent with our theoretical results. These represent situations where the contamination occurs as a small-sample problem. In the next section, we show that when the contamination holds asymptotically then the size distortions and power problems can be even more severe.

### 4.3 General Low Frequency Contamination

We now discuss statistical environments and HAR inference tests for which the low frequency contamination results of Section 3 hold even asymptotically. This means that $d^* > 0$ for all $T$ and as $T \to \infty$. This comprises the class of HAR tests that admit a nonstationary alternative hypotheses. This class is very large and include most HAR-based tests. Examples include tests for forecast evaluation [e.g., Casini (2018), Diebold and Mariano (1995), Giacomini and Rossi (2009, 2010), Giacomini and White (2006), Perron and Yamamoto (2021) and West (1996)], tests for structural changes [e.g., Andrews (1993), Bai and Perron (1998), Casini and Perron (2020b, 2020c, 2020d), Elliott and Müller (2007), and Qu and Perron (2007)], tests for time-varying parameters [e.g., Cai (2007) and Chen and Hong (2012)], tests for regime switching [e.g., Hamilton (1989) and
Qu and Zhuo (2020)] and many others.\(^9\) Here we consider the Diebold-Mariano test for the sake of illustration and remark that similar issues apply to the other HAR tests mentioned above.

The Diebold-Mariano test statistic is defined as \(t_{DM} \triangleq T_n^{1/2} \delta L / \sqrt{J_{dL,T}}\), where \(\delta L\) is the average of the loss differentials between two competing forecast models, \(J_{dL,T}\) is an estimate of the LRV of the the loss differential series and \(T_n\) is the number of observations in the out-of-sample. We use the quadratic loss. We consider an out-of-sample forecasting exercise with a fixed forecasting scheme where, given a sample of \(T\) observations, \(0.5T\) observations are used for the in-sample and the remaining half is used for prediction [see Perron and Yamamoto (2021) for recommendations on using a fixed scheme in the presence of breaks]. The DGP under the null hypotheses is given by \(y_t = 1 + \beta_0 x_{t-1}^{(0)} + e_t\) where \(x_{t-1}^{(0)}\) is i.i.d. \(\mathcal{N}(1,1)\), \(e_t = 0.3e_{t-1} + u_t\) with \(u_t\) is i.i.d. \(\mathcal{N}(0,1)\), and we set \(\beta_0 = 1\) and \(T = 400\). The two competing models both involve an intercept but differ on the predictor used in place of \(x_t^{(0)}\). The first forecast model uses \(x_t^{(1)}\) while the second uses \(x_t^{(2)}\) where \(x_t^{(1)}\) and \(x_t^{(2)}\) are independent i.i.d. \(\mathcal{N}(1, 1)\) sequences, both independent from \(x_t^{(0)}\). Each forecast model generates a sequence of \(\tau (= 1)\)-step ahead out-of-sample losses \(L_t^{(j)}\) \((j = 1, 2)\) for \(t = T/2 + 1, \ldots, T - \tau\). Then \(d_t \triangleq L_t^{(2)} - L_t^{(1)}\) denotes the loss differential at time \(t\). The Diebold-Mariano test rejects the null hypotheses of equal predictive ability when \(\delta L\) is sufficiently far from zero. Under the alternative hypothesis, the two competing forecast models are as follows: the first uses \(x_t^{(1)} = x_t^{(0)} + u_{X_1,t}\) where \(u_{X_1,t}\) is i.i.d. \(\mathcal{N}(0,1)\) while the second uses \(x_t^{(2)} = x_t^{(0)} + 0.2z_t + 2u_{X_2,t}\) for \(t \in [1, \ldots, 3T/4 - 1, 3T/4 + 21, \ldots] T\) and \(x_t^{(2)} = \delta (t/T) + 0.2z_t + 2u_{X_3,t}\) for \(t = 3T/4, \ldots, 3T/4 + 20\) with \(u_{X_3,t}\) is i.i.d. \(\mathcal{N}(0,1)\), where \(z_t\) has the same distribution as \(x_t^{(0)}\).

We consider four specifications for \(\delta(\cdot)\). In the first \(x_t^{(2)}\) is subject to an abrupt break in the mean \(\delta (t/T) = \delta > 0\), in the second \(x_t^{(2)}\) is locally stationary with time-varying mean \(\delta (t/T) = \delta (\sin(t/T - 3/4))\), in the third specification \(x_t^{(2)} = x_t^{(0)} + 0.2z_t + 2u_{X_3,t}\) for \(t \in [1, \ldots, T/2 - 30, T/2 + 21, \ldots] T\) and \(x_t^{(2)} = \delta (t/T) + 0.2z_t + 2u_{X_3,t}\) for \(t = T/2 - 30, \ldots, T/2 + 20\) with \(\delta (t/T) = \delta (\sin(t/T - 1/2 - 30/T))\), in the fourth \(x_t^{(2)}\) is the same as in the second with in addition two outliers \(x_t^{(2)} \sim \text{Uniform}(\lceil c \rceil, 5 |c|)\) for \(t = 6T/10, 8T/10\) where \(c = -1/\left(\sqrt{2\text{erfc}^{-1}(3/2)}\right)\) \(\text{med}(|x_t^{(2)} - \text{med}(x_t^{(2)}))\) where \(x^{(2)}_{t_1} = \left(x_t^{(2)} |_{t_1}^T\right)\). That is, in the second model \(x_t^{(2)}\) is locally stationary only in the out-of-sample, in the third it is locally stationary in both the in-sample and out-of-sample and in the fourth model \(x_t^{(2)}\) has two outliers in the out-of-sample. The location of the outliers is irrelevant.

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\(^9\)In some cases the low frequency contamination could be reduced if one uses a test that accounts properly for the form of nonstationary under the alternative hypotheses. For example, consider testing for breaks. Suppose that there are two breaks and one first uses a test for one break versus no break and then a test of two breaks versus one break if the first test rejects the null hypotheses of no break, and so on. The test in the first step would suffer from low frequency contamination since under the alternative of one break the estimate of the LRV is contaminated by the presence of the second break. Thus, one would conclude that there is no break. In contrast, if one uses a test that allows for an unknown number of breaks given some upper bound [i.e., the UDMax test of Bai and Perron (1998)], the contamination can be reduced because this test would account for the correct form of nonstationarity. However, if instead of two breaks the true model involves other forms of nonstationarity (e.g., slowly-varying parameters, smooth breaks, etc.) then also the UDMax test would suffer from low frequency contamination.
for the results; they can also occur in the in-sample.

Table 6 reports the size and the power for all models. We begin with the case $\delta(t/T) = \delta > 0$ (top panel). The size of the test using the DK-HAC estimators is accurate while the test using other LRV estimators are oversized with the exception of the KVB’s fixed-$b$ method for which the rejection rate is equal to zero. The HAR tests using existing LRV estimators have lower power relative to that obtained with the DK-HAC estimators for small values of $\delta$. When $\delta$ increases the tests standardized by the HAC estimators of Andrews (1991) and Newey and West (1987), and by the KVB’s fixed-$b$ and EWC LRV estimators display non-monotonic power gradually converging to zero. In contrast, when using the DK-HAC estimators the test has monotonic power that reaches and maintains unit power. The results for the other models are even stronger. In general, except for the DK-HAC estimators, all other tests display serious power problems. Thus, either form of nonstationarity or outliers leads to similar implications, consistent with our theoretical results.

In order to further assess the theoretical results from Section 3, Figure 3-5 report the plots of $d_t$, its sample autocovariances and its periodogram, for $\delta = 1, 2, 5$, respectively. We only consider the case $\delta_t = \delta > 0$. The other cases lead to the same conclusions. For $\delta = 1$, Figure 3 (mid panel) shows that $\hat{\Gamma}(k)$ decays slowly. As $\delta$ increases, Figure 4-5 (mid panels), $\hat{\Gamma}(k)$ decays even more slowly at a rate far from the typical exponential decay of short memory processes. It shows a power-like decay that is statistically significant at the 10% significance level up to lag 80. This suggests evidence of long memory. However, the data are short memory with small temporal dependence. What is generating the spurious long memory effect is the nonstationarity present under the alternative hypotheses. This is visible in the top panels which present plots of $d_t$ for the first specification. The shift in the mean of $d_t$ for $t = 3T/4, \ldots, 3T/4 + 20$ is responsible for the long memory effect. This corresponds to the second term of (3.3) in Theorem 3.1. The negative autocovariances follow from the behavior of $d_t$ for observations $t \neq 3T/4, \ldots, 3T/4 + 20$. That is, a positive $d_t$ at time $t$ corresponds to $x_{t-1}^{(2)}$ predicting $y_t$ worse than what $x_{t-1}^{(1)}$ does and this is likely to be followed by $x_{t}^{(2)}$ predicting $y_{t+1}$ better than what $x_{t}^{(1)}$ does. Thus, the overall behavior of the sample autocovariance is as predicted by Theorem 3.1. For small lags, $\hat{\Gamma}(k)$ shows a power-like decay and it is positive. As $k$ increases to medium lags, the autocovariances turn negative. However, $d^*$ in (3.3) makes these autocovariances closer to zero since $d^* > 0$. For large $k$, $\hat{\Gamma}(k)$ approaches zero. Next, we move to the bottom panels which plot the periodogram of $\{d_t\}$. It is unbounded at frequencies close to $\omega = 0$ as predicted by Theorem 3.2 and as would occur if long memory was present. It also explains why the Diebold-Mariano test normalized by Newey-West’s, Andrews’, KVB’s fixed-$b$ and EWC’s LRV estimators have serious power problems. These LRV estimators are inflated and consequently the tests lose power. The figures show that as we raise $\delta$ the more severe these issues and the power losses so that the power eventually reaches zero. This is consistent with our theory since $d^*$ is increasing in $\delta$ (cf. $d^* \approx 0.1 \cdot 0.9\delta^2$).

We now verify the results about the local sample autocovariance $\hat{c}_T(u, k)$ and the correspon-
ding local periodogram from Theorem 3.3-3.4. We set $n_{2,T} = T^{0.6} = 36$ following the MSE criterion of Casini (2021). We consider (i) $u = 236/T$, (ii-a) $u = T_1^0/T = 3/4$ and (ii-b) $u = 264/T$. Note that cases (i)-(ii-b) correspond to parts (i)-(ii-b) in Theorem 3.3-3.4. We consider $\delta = 1, 2$ and $5$. According to Theorem 3.3-3.4, we should expect long memory features only for case (ii-a). Figures 6-11 confirm this. The results pertaining to case (ii-a) are plotted in the middle panels. Figures 6, 8 and 10 show that the local autocovariance displays slow decay similar to the pattern discussed above for the sample autocovariance $\tilde{\Gamma}(k)$ and that this problem becomes more severe as $\delta$ increases. Such long memory features also appear for $I_L(3/4, \omega)$. The middle panels in Figure 7, 9 and 11 show that the local periodogram at $u = 3/4$ and at a frequency close to $\omega = 0$ are extremely large. The latter result is consistent with Theorem 3.4-(ii-a) which suggests that $I_L,T(3/4, \omega) \to \infty$ as $\omega \to 0$. For case (i) and (ii-b) both figures show that the local autocovariance and the local periodogram do not display long memory features. Indeed, they have forms similar to those of a short memory process, a result consistent with Theorem 3.3-3.4 also for cases (i) and (ii-b).

It is interesting to explain why HAR inference based on the DK-HAC estimators does not suffer from the low frequency contamination even if case (ii-a) occurs. Note that the DK-HAC estimator computes an average of the local spectral density over time blocks. If it happens that one of these blocks contains a discontinuity in the spectrum, then as in case (ii-a) some bias would arise for the local spectral density estimate corresponding to that block. However, by virtue of the time-averaging over blocks that bias becomes negligible. Hence, nonparametric smoothing over time asymptotically cancels the bias, so that inference based on the DK-HAC estimators is robust to nonstationarity.

4.4 Discussion

In summary, the sample autocovariance and the periodogram are sensitive to nonstationarity in that they may display characteristics typical of long memory even if the data are short memory. We refer to this phenomenon as low frequency contamination induced by unaccounted nonstationarity. In some situations these issues only imply a small-sample problem [cf. Section 4.2]. In others, they have an effect even asymptotically. In either case, the theory in Section 3 provides useful guidance about the properties of the sample autocovariance and the periodogram in finite-samples. It also provides accurate approximations for misspecified models or models with outliers as discussed above. Since LRV estimates are direct inputs for inference methods in the context of potentially autocorrelated data, our theory offers new insights for the properties of HAR inference tests. The use of HAC standard errors has become the standard practice, and recent theoretical developments in HAR inference advocated the use of fixed-$b$ or long bandwidth LRV estimators on the basis that they offer better size control when there is strong dependence in the data. Our results suggest that care is needed before applying such methods because they are highly sensitive to effects akin
to long memory arising from nonstationarity. The concern is then that the use of long bandwidths lead to overestimation of the true LRV due to low frequency contamination. Consequently, HAR tests can lose power dramatically, a problem that occurs also for the classical HAC estimators, though to a lesser extent since they use a smaller number of sample autocovariances.

Our theory also suggests a solution to this problem. This entails the use of nonparametric smoothing over time which avoids combining observations that belong to different regimes. This accounts for nonstationarity and prevents spurious long memory effects. An exception where some bias may arise for $\hat{c}_T(u, k)$ and $I_T(u, \omega)$ is when a discontinuity in the spectrum falls close to $\lfloor Tu \rfloor$. However, this problem is simple to address because one can use a pre-test for discontinuities in the spectrum and exclude those $u \in (0, 1)$ that are close to a discontinuity in the spectrum. Casini and Perron (2021a) proposed such a test and used it for the DK-HAC estimators which were shown to be robust to such cases.

5 Conclusions

Economic time series are highly nonstationary and models might be misspecified. If nonstationary is not accounted for properly, parameter estimates and, in particular, asymptotic variance estimates can be largely biased. We establish results on the low frequency contamination induced by nonstationarity and misspecification for the sample autocovariance and the periodogram. These estimates can exhibit features akin to long memory when the data are nonstationary short memory. We distinguish cases where this contamination only implies a small-sample problem and cases where the problem remains asymptotically. We propose a solution to this problem based on nonparametric smoothing which is shown, using theoretical arguments, to be robust. Since the autocovariances and the periodogram are basic elements for heteroskedasticity and autocorrelation robust (HAR) inference, our results provide insights on the important debate between consistent versus inconsistent LRV estimation. Indeed, the properties of long bandwidths/fixed-$b$ methods are only known under stationarity. Our results show that existing LRV estimators tend to be inflated when the data are nonstationarity. This results in HAR tests that can be undersized and exhibit dramatic power losses or even no power. Long bandwidths/fixed-$b$ HAR tests suffer more from low frequency contamination relative to HAR tests based on HAC estimators, whereas the DK-HAC estimators do not suffer from this problem.
References


CHAN, K. W. (2020): “Mean-Structure and Autocorrelation Consistent Covariance Matrix Esti-
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6 Appendix

Figure 1: Plot of $V_t$ for model M1. The sample size is $T = 200$. $T^0_1 = 20$. Also reported in red dashed lines are the sample averages in the two regimes with $\bar{V}_1 = 1.27$ and $\bar{V}_2 = -0.03$. 
Figure 2: ACF of $V_{1,t}$ (top panel), ACF of $V_{2,t}$ (mid panel) and ACF of $V_t$ (bottom panel) for model M1.
Figure 3: a) top panel: plot of $\{d_t\}$; b) mid-panel: plot of the sample autocovariances $\hat{\Gamma}(k)$ of $\{d_t\}$; c) bottom panel: plot the periodogram $I(\omega)$ of $\{d_t\}$. In all panels $\delta = 1$. 

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Figure 4: a) top panel: plot of \( \{ d_t \} \); b) mid-panel: plot of the sample autocovariances \( \hat{\Gamma}(k) \) of \( \{ d_t \} \); c) bottom panel: plot the periodogram \( I(\omega) \) of \( \{ d_t \} \). In all panels \( \delta = 2 \).
Figure 5: a) top panel: plot of \( \{d_t\} \); b) mid-panel: plot of the sample autocovariances \( \hat{\Gamma}(k) \) of \( \{d_t\} \); c) bottom panel: plot the periodogram \( I(\omega) \) of \( \{d_t\} \). In all panels \( \delta = 5 \).
Figure 6: The figure plots $\hat{c}(u, k)$ for $u = 236/400, 264/400$ in the top, mid and bottom panel, respectively. In all panels $\delta = 1$. 

Local autocovariance $\hat{c}(236/400, k)$; $\delta = 1$

Local autocovariance $\hat{c}(300/400, k)$; $\delta = 1$

Local autocovariance $\hat{c}(264/400, k)$; $\delta = 1$
Figure 7: The figure plots $I_L(u, \omega)$ for $u = \frac{236}{400}, \lambda^0_1$, $\frac{264}{400}$ in the top, mid and bottom panel, respectively. In all panels $\delta = 1$. 
Figure 8: The figure plots $\tilde{c}_T(u, k)$ for $u = 236/400, \lambda_1^0, 264/400$ in the top, mid and bottom panel, respectively. In all panels $\delta = 2$. 

Local autocovariance $\tilde{c}(236/400, k); \delta = 2$

Local autocovariance $\tilde{c}(300/400, k); \delta = 2$

Local autocovariance $\tilde{c}(264/400, k); \delta = 2$
Figure 9: The figure plots $I_L(u, \omega)$ for $u = 236/400, \lambda^0_1, 264/400$ in the top, mid and bottom panel, respectively. In all panels $\delta = 2$. 
Figure 10: The figure plots \( \hat{c}_T(u, k) \) for \( u = 236/400, \lambda_1^0, 264/400 \) in the top, mid and bottom panel, respectively. In all panels \( \delta = 5 \).
Figure 11: The figure plots $I_L(u, \omega)$ for $u = 236/400, \lambda_1^0, 264/400$ in the top, mid and bottom panel, respectively. In all panels $\delta = 5$. 
LOW FREQUENCY CONTAMINATION IN HAR INFERENCE

Table 1: Comparison between the theoretical autocovariance and the sample estimates

<table>
<thead>
<tr>
<th>$k$</th>
<th>$\Gamma_T(k)$</th>
<th>$\hat{\Gamma}(k)$</th>
<th>$\hat{\Gamma}(k) - \hat{d^*}$</th>
<th>$\Gamma_{DK}(k)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.37</td>
<td>0.54</td>
<td>0.37</td>
<td>0.63</td>
</tr>
<tr>
<td>1</td>
<td>0.13</td>
<td>0.29</td>
<td>0.13</td>
<td>0.25</td>
</tr>
<tr>
<td>2</td>
<td>0.01</td>
<td>0.16</td>
<td>0.01</td>
<td>0.03</td>
</tr>
<tr>
<td>5</td>
<td>0.00</td>
<td>0.05</td>
<td>-0.03</td>
<td>0.00</td>
</tr>
<tr>
<td>10</td>
<td>-0.03</td>
<td>0.02</td>
<td>-0.07</td>
<td>0.00</td>
</tr>
</tbody>
</table>

The table reports the values of a) the theoretical autocovariance $\Gamma_T(k)$; b) the sample autocovariance $\hat{\Gamma}(k)$; c) the corrected sample autocovariance $\hat{\Gamma}(k) - \hat{d^*}$; d) the double kernel sample autocovariance $\Gamma_{DK}(k)$, for $k = 0, 1, 2, 5$ and 10, with the process $\{V_t\}$ defined in Section 4.2 with $T = 200$.

Table 2: Empirical small-sample size and power of $t$-test for model M1

<table>
<thead>
<tr>
<th>$\alpha = 0.05$, $T = 200$</th>
<th>$\delta = 0$ (size)</th>
<th>$\delta = 0.05$</th>
<th>$\delta = 0.1$</th>
<th>$\delta = 0.15$</th>
<th>$\delta = 0.25$</th>
<th>$\delta = 1$</th>
<th>$\delta = 1.5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$J_{DK,T}$</td>
<td>0.068</td>
<td>0.189</td>
<td>0.286</td>
<td>0.460</td>
<td>0.661</td>
<td>0.992</td>
<td>1.000</td>
</tr>
<tr>
<td>$J_{DK,pw,SLS,T}$</td>
<td>0.045</td>
<td>0.085</td>
<td>0.199</td>
<td>0.332</td>
<td>0.612</td>
<td>0.976</td>
<td>1.000</td>
</tr>
<tr>
<td>$J_{DK,pw,SLS,\mu,T}$</td>
<td>0.046</td>
<td>0.090</td>
<td>0.202</td>
<td>0.333</td>
<td>0.613</td>
<td>0.977</td>
<td>1.000</td>
</tr>
<tr>
<td>Andrews (1991)</td>
<td>0.039</td>
<td>0.095</td>
<td>0.185</td>
<td>0.383</td>
<td>0.623</td>
<td>0.968</td>
<td>0.999</td>
</tr>
<tr>
<td>Andrews (1991), prewhite</td>
<td>0.115</td>
<td>0.168</td>
<td>0.304</td>
<td>0.447</td>
<td>0.650</td>
<td>0.988</td>
<td>0.999</td>
</tr>
<tr>
<td>Newey-West (1987)</td>
<td>0.209</td>
<td>0.272</td>
<td>0.398</td>
<td>0.516</td>
<td>0.689</td>
<td>0.997</td>
<td>1.000</td>
</tr>
<tr>
<td>Newey-West (1987), fixed-$b$ (KVB)</td>
<td>0.004</td>
<td>0.018</td>
<td>0.063</td>
<td>0.139</td>
<td>0.301</td>
<td>0.870</td>
<td>0.969</td>
</tr>
<tr>
<td>EWC</td>
<td>0.011</td>
<td>0.038</td>
<td>0.137</td>
<td>0.273</td>
<td>0.539</td>
<td>0.978</td>
<td>0.999</td>
</tr>
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</table>

Table 3: Empirical small-sample size and power of the of $t$-test for model M2

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<th>$\alpha = 0.05$, $T = 200$</th>
<th>$\delta = 0$ (size)</th>
<th>$\delta = 0.15$</th>
<th>$\delta = 0.2$</th>
<th>$\delta = 0.25$</th>
<th>$\delta = 0.3$</th>
<th>$\delta = 0.5$</th>
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<tr>
<td>$J_{DK,T}$</td>
<td>0.059</td>
<td>0.415</td>
<td>0.815</td>
<td>0.974</td>
<td>0.974</td>
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<tr>
<td>$J_{DK,pw,SLS,T}$</td>
<td>0.058</td>
<td>0.262</td>
<td>0.632</td>
<td>0.899</td>
<td>0.899</td>
<td>1.000</td>
</tr>
<tr>
<td>$J_{DK,pw,SLS,\mu,T}$</td>
<td>0.053</td>
<td>0.246</td>
<td>0.616</td>
<td>0.894</td>
<td>0.894</td>
<td>1.000</td>
</tr>
<tr>
<td>Andrews (1991)</td>
<td>0.064</td>
<td>0.228</td>
<td>0.564</td>
<td>0.892</td>
<td>0.830</td>
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<tr>
<td>Andrews (1991), prewhite</td>
<td>0.252</td>
<td>0.564</td>
<td>0.904</td>
<td>0.992</td>
<td>0.991</td>
<td>1.000</td>
</tr>
<tr>
<td>Newey-West (1987)</td>
<td>0.133</td>
<td>0.388</td>
<td>0.821</td>
<td>0.981</td>
<td>0.971</td>
<td>1.000</td>
</tr>
<tr>
<td>Newey-West (1987), fixed-$b$ (KVB)</td>
<td>0.000</td>
<td>0.077</td>
<td>0.018</td>
<td>0.356</td>
<td>0.356</td>
<td>0.971</td>
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<tr>
<td>EWC</td>
<td>0.004</td>
<td>0.045</td>
<td>0.255</td>
<td>0.632</td>
<td>0.637</td>
<td>1.000</td>
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Table 5: Empirical small-sample size and power of the of \( t \)-test for model M4

\[ \alpha = 0.05, \ T = 200 \]

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<tr>
<td>( \delta = 0 ) (size)</td>
<td>( \delta = 0.1 )</td>
<td>( \delta = 0.15 )</td>
<td>( \delta = 0.2 )</td>
<td>( \delta = 0.25 )</td>
<td>( \delta = 0.5 )</td>
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<tr>
<td>0.067</td>
<td>0.558</td>
<td>0.748</td>
<td>0.870</td>
<td>0.945</td>
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<td>0.065</td>
<td>0.301</td>
<td>0.495</td>
<td>0.618</td>
<td>0.736</td>
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<td>0.037</td>
<td>0.351</td>
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<td>0.656</td>
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<td>0.016</td>
<td>0.253</td>
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<td>0.916</td>
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<td>0.346</td>
<td>0.954</td>
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<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.005</td>
<td>0.015</td>
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<td>0.024</td>
<td>0.240</td>
<td>0.486</td>
<td>0.596</td>
<td>0.681</td>
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Table 4: Empirical small-sample size and power of the of \( t \)-test for model M3

\[ \alpha = 0.05, \ T = 200 \]

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<tbody>
<tr>
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<td>( \delta = 0.1 )</td>
<td>( \delta = 0.15 )</td>
<td>( \delta = 0.2 )</td>
<td>( \delta = 0.25 )</td>
<td>( \delta = 0.5 )</td>
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<tr>
<td>0.086</td>
<td>0.552</td>
<td>0.930</td>
<td>0.992</td>
<td>1.000</td>
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</tr>
<tr>
<td>0.065</td>
<td>0.436</td>
<td>0.887</td>
<td>0.971</td>
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<td>0.063</td>
<td>0.415</td>
<td>0.875</td>
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<tr>
<td>0.017</td>
<td>0.257</td>
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<td>0.036</td>
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<td>0.000</td>
<td>0.051</td>
<td>0.299</td>
<td>0.699</td>
<td>0.937</td>
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<td>Table 6: Empirical small-sample size and power of the DM (1995) test</td>
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<tr>
<td>( \alpha = 0.05, T = 200 )</td>
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<tr>
<td>( \delta &gt; 0 )</td>
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<tr>
<td>( \delta = 0 ) (size) ( \delta = 0.2 ) ( \delta = 0.5 ) ( \delta = 2 ) ( \delta = 5 ) ( \delta = 10 )</td>
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<td></td>
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</tr>
<tr>
<td>( J_{DK,T} )</td>
<td>0.033</td>
<td>0.312</td>
<td>0.551</td>
<td>0.997</td>
<td>1.000</td>
<td>1.000</td>
<td></td>
</tr>
<tr>
<td>( \hat{J}_{DK,pw,SLS,T} )</td>
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<td>0.401</td>
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<tr>
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<td>0.252</td>
<td>0.268</td>
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<tr>
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<td>0.987</td>
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<tr>
<td>Andrews (1991)</td>
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<td>0.720</td>
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<tr>
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7 Mathematical Appendix

7.1 Proof of Theorem 3.1

Let \( X_j = (Tr_j)^{-1} \sum_{t=[T\lambda_j^{-1}]}^{[T\lambda_j]} X_t \), \( \mu_{2,j}(u) = E(X_{uT})E(X_{uT}) \) for \( T_{j-1}^0 \leq Tu \leq T_j^0 \) and \( \widehat{\mu}_{2,j} = r_j^{-1} \int_{\lambda_j^{-1}}^{\lambda_j} \mu_{2,j}(u) du \). By Assumption 2.1-2.2-(i), the latter implying ergodicity, it follows that for fixed \( k \geq 0 \) that

\[
\hat{\Gamma}(k) = \sum_{j=1}^{m_0+1} r_j \frac{1}{Tr_j} \sum_{t=[T\lambda_j^{-1}]}^{[T\lambda_j]} X_t X_{t-k} - \left( \sum_{j=1}^{m_0+1} r_j \frac{1}{Tr_j} \sum_{t=[T\lambda_j^{-1}]}^{[T\lambda_j]} X_t \right)^2 + o_{a.s.} (1)
\]

\[
= \sum_{j=1}^{m_0+1} \int_{\lambda_j^{-1}}^{\lambda_j} c(u, k) du + \sum_{j=1}^{m_0+1} r_j \frac{1}{Tr_j} \sum_{t=[T\lambda_j^{-1}]}^{[T\lambda_j]} \mathbb{E}(X_t) \mathbb{E}(X_{t-k}) - \left( \sum_{j=1}^{m_0+1} r_j \mathbb{X}_j \right)^2 + O(T^{-1}) + o_{a.s.} (1)
\]

where we have used \( \mathbb{E}(X_{t-k}) - \mathbb{E}(X_t) = O(k/T) \) by local stationarity in the third equality. Note that by ergodicity and an approximation to Riemann sums, we have

\[
\sum_{j=1}^{m_0+1} r_j \mathbb{X}_j - \sum_{j=1}^{m_0+1} r_j \overline{\mathbb{X}}_j = \sum_{j=1}^{m_0+1} r_j \mathbb{E}(X_j) + \sum_{j=1}^{m_0+1} r_j \mathbb{E}(X_j) - \sum_{j=1}^{m_0+1} r_j \overline{\mathbb{X}}_j = o_{a.s.} (1) + O(T^{-1}) \tag{7.1}
\]

Basic manipulations show that

\[
\sum_{j \neq j_1} r_{j_1} r_{j_2} (\overline{\mathbb{X}}_{j_2} - \overline{\mathbb{X}}_{j_1})^2
\]

\[
= \sum_{j \neq j_1} r_{j_1} r_{j_2} (\overline{\mathbb{X}}_{j_2}^2 + \overline{\mathbb{X}}_{j_1}^2 - 2 \overline{\mathbb{X}}_{j_2} \overline{\mathbb{X}}_{j_1})
\]

\[
= \sum_{1 \leq j_2 \leq m_0+1} r_{j_2} \overline{\mathbb{X}}_{j_2}^2 (1 - r_{j_2}) + \sum_{1 \leq j_1 \leq m_0+1} r_{j_1} \overline{\mathbb{X}}_{j_1}^2 (1 - r_{j_1}) - 2 \sum_{j_1 \neq j_2} r_{j_1} r_{j_2} \overline{\mathbb{X}}_{j_2} \overline{\mathbb{X}}_{j_1}
\]
\[ (Tr_j - k) \sum_{t = [T\lambda^0_j - 1] + 1}^{[T\lambda^0_j]} \mu^2(t/T) \geq \left( \sum_{t = [T\lambda^0_j - 1] + 1}^{[T\lambda^0_j]} \mu(t/T) \right)^2. \]  

(7.3)

Thus,

\[ \sum_{j=1}^{m_0+1} \frac{1}{Tr_j} \sum_{t = [T\lambda^0_j - 1] + 1}^{[T\lambda^0_j]} \mu^2(t/T) = \sum_{j=1}^{m_0+1} \frac{1}{Tr_j (Tr_j - k)} (Tr_j - k) \sum_{t = [T\lambda^0_j - 1] + 1}^{[T\lambda^0_j]} \mu^2(t/T) \]

\[ \geq \sum_{j=1}^{m_0+1} \frac{1}{Tr_j (Tr_j - k)} \left( \sum_{t = [T\lambda^0_j - 1] + 1}^{[T\lambda^0_j]} \mu(t/T) \right)^2 \]

\[ = \sum_{1 \leq j \leq m_0+1} r_j \bar{\mu}_j + o(1). \]  

(7.4)

Using (7.1)-(7.4) we have,

\[ \hat{\Gamma}(k) = \int_0^1 c(u, k) du + \sum_{j=1}^{m_0+1} \frac{1}{Tr_j} \sum_{t = [T\lambda^0_j - 1] + 1}^{[T\lambda^0_j]} \mu^2(t/T) - \left( \sum_{j=1}^{m_0+1} r_j \bar{X}_j \right)^2 + o_{a.s.} (1) \]

\[ \geq \int_0^1 c(u, k) du + \sum_{j=1}^{m_0+1} r_j \bar{\mu}_{2j} - \left( \sum_{j=1}^{m_0+1} r_j \bar{X}_j \right)^2 + O \left( T^{-1} \right) + o_{a.s.} (1) \]

\[ = \int_0^1 c(u, k) du + 2^{-1} \sum_{j_1 \neq j_2} r_{j_1} r_{j_2} (\bar{\mu}_{j_2} - \bar{\mu}_{j_1})^2 + O \left( T^{-1} \right) + o_{a.s.} (1). \]  

(7.5)

The claim that \( \hat{\Gamma}(k) \geq d \) \( \mathbb{P} \)-a.s. as \( k \to \infty \) follows from Assumption 2.2-(i) since this implies that \( c(u, k) \to 0 \) as \( k \to \infty \) and from the fact that the second term on the right-hand side of (7.5) does not depend on \( k \). If in addition it holds that \( \mu_j(t/T) = \mu_j \) for \( j = 1, \ldots, m_0 + 1 \), then (7.3) holds with equality and the result follows as a special case of (7.5). \( \square \)

### 7.2 Proof of Theorem 3.2

**Lemma 7.1.** Assume that \( \{V_{t,T}\} \) satisfies Definition 2.1. Under Assumption 2.1-2.2 and 3.1-(ii),

\[ \sum_{j_1 \neq j_2} \frac{1}{T} \sum_{t = [T\lambda^0_{j_1} - 1] + 1}^{[T\lambda^0_{j_2}]} \sum_{s = [T\lambda^0_{j_2} - 1] + 1}^{[T\lambda^0_{j_1}]} \mathbb{E} \left( (X_t - \mu(t/T)) (X_s - \mu(s/T)) \right) \exp \left( -i \omega_l (t - s) \right) = o(1). \]

**Proof.** Let \( \tau_{j_1,j_2} = \max \{r_{j_1}, r_{j_2}\} \) and \( \tau_{j_1,j_2} = \min \{r_{j_1}, r_{j_2}\} \). We consider the case of adjacent regimes (i.e., \( j_2 = j_1 + 1 \)) which also provides an upper bound for non-adjacent regimes due to the short memory property. For any \( k = s - t = 1, \ldots, \lfloor T\lambda_{j_1,j_2} \rfloor \) there are \( k \) pairs in the above sum. The double sum above
(over $t$ and $s$) can be split into

$$
T^{-1} \left| \sum_{k=1}^{[CT^n]} \Gamma_{\{1:[CT^n]\}} (\cdot, k) \right| + T^{-1} \left| \sum_{k=[CT^n]+1}^{[hT]} \Gamma_{\{[CT^n]+1:[hT]\}} (\cdot, k) \right| \\
+ T^{-1} \left| \sum_{k=[hT]+1}^{[T_{\tilde{z}_{j_1}},j_2]-1} \Gamma_{\{[hT]+1:[T_{\tilde{z}_{j_1}},j_2]-1\}} (\cdot, k) \right| + T^{-1} \left| \sum_{k=[T_{\tilde{z}_{j_1}},j_2]} \Gamma_{\{T_{\tilde{z}_{j_1}},j_2}:\tilde{z}_{j_1},j_2\}} (\cdot, k) \right|
$$

where $C > 0$, $0 < h < 1$ with $[hT] < [T_{\tilde{z}_{j_1}},j_2] - 1$, and $\Gamma_S (\cdot, k)$ is the sum of the autocovariances at lag $k$ computed at the time points corresponding to $k \in S$. Note that the term $|\exp (-i\omega l (\pm k))|$ can be bounded by some constant. The sums run over only $k > 0$ because by symmetry $\Gamma_u (k) = \Gamma_{u-k/T} (-k)$. Consider the first sum in (7.6). This is of order $O (T^{-1}T^{2\kappa})$ which goes to zero given $\kappa < 1/2$. The second sum is also negligible using the following arguments. By Assumption 3.1-(ii), $\Gamma (u, k) = C_{u,k}k^{-m}$ with $m > 2$ and choosing $C$ large enough yields that the second sum of (7.6) converges to zero. In the third sum, the number of summands grows at rate $O (T)$ and for each lag $k$ there are $O (T)$ autocovariances. However, by Assumption 3.1-(ii) each autocovariance is $O (T^{-m})$. Thus, the bound is $O (T^{-1}T^{2-m})$ which goes to zero as $T \to \infty$. The difference between the arguments used for the third sum and fourth sums is that now we do not have $O (T)$ autocovariances for each lag $k$. Thus, the bound for the fourth sum cannot be greater than the bound for the third sum. Thus, the fourth sum also converges to zero. \( \square \)

**Proof of Theorem 3.2.** We have,

$$
I_T (\omega_l) = \left| \frac{1}{\sqrt{T}} \sum_{j=1}^{m_0+1} \sum_{t=[T\lambda_{j-1}^0]+1}^{[T\lambda_j^0]} \exp (-i\omega_l t) X_t \right|^2
$$

$$
= \left| \frac{1}{\sqrt{T}} \sum_{j=1}^{m_0+1} \sum_{t=[T\lambda_{j-1}^0]+1}^{[T\lambda_j^0]} (X_t - \mu (t/T)) \exp (-i\omega_l t) + \frac{1}{\sqrt{T}} \sum_{j=1}^{m_0+1} \sum_{t=[T\lambda_{j-1}^0]+1}^{[T\lambda_j^0]} \mu (t/T) \exp (-i\omega_l t) \right|^2.
$$

From Assumption 3.1,

$$
\left| \sum_{j=1}^{m_0+1} \sum_{t=[T\lambda_{j-1}^0]+1}^{[T\lambda_j^0]} \mu (t/T) \exp (-i\omega_l t) \right|^2
$$

$$
\geq \left| \sum_{j=1}^{m_0+1} B_j \sum_{t=[T\lambda_{j-1}^0]+1}^{[T\lambda_j^0]} \exp (-i\omega_l t) \right|^2
$$

$$
= \left| \sum_{j=1}^{m_0+1} B_j \exp (-i\omega_l ([T\lambda_{j-1}^0] + 1)) \sum_{t=0}^{[T\lambda_j^0] - [T\lambda_{j-1}^0] - 1} \exp (-i\omega_l t) \right|^2
$$

$$
= \left| \exp (-i\omega_l) \sum_{j=1}^{m_0+1} B_j \exp (-i\omega_l ([T\lambda_{j-1}^0])) \left( 1 - \exp (-i\omega_l ([T\lambda_j^0] - [T\lambda_{j-1}^0])) \right) \right|^2
$$

$$
= \left| \frac{\exp (-i\omega_l)}{1 - \exp (-i\omega_l)} \sum_{j=1}^{m_0+1} B_j \exp (-i\omega_l ([T\lambda_{j-1}^0])) - \exp (-i\omega_l [T\lambda_j^0]) \right|^2,
$$

A-3
By Lemma 7.1, it is sufficient to consider the cross-products within each regime $j$,

$$
\frac{\exp(-i\omega j)}{1 - \exp(-i\omega j)} \sum_{j=1}^{m_0+1} B_j \left( \exp \left( -i\omega \left( \lfloor T \lambda_{j-1}^0 \rfloor \right) \right) - \exp \left( -i\omega \lfloor T \lambda_j^0 \rfloor \right) \right)
$$

$$
= \frac{\exp(-i\omega j)}{1 - \exp(-i\omega j)} \left[ B_1 - B_{m_0+1} - \sum_{j=1}^{m_0} (B_j - B_{j-1}) \exp \left( -i\omega \lfloor T \lambda_j^0 \rfloor \right) \right].
$$

By Lemma 7.1, it is sufficient to consider the cross-products within each regime $j$,

$$
\mathbb{E} \left( I_T (\omega_l) \right) \geq \sum_{j=1}^{m_0+1} \frac{r_j}{T r_j} \mathbb{E} \left[ \sum_{t=\lfloor T \lambda_{j-1}^0 \rfloor +1}^{\lfloor T \lambda_j^0 \rfloor} \sum_{s=\lfloor T \lambda_{j-1}^0 \rfloor +1}^{\lfloor T \lambda_j^0 \rfloor} (X_t - \mu (t/T)) (X_s - \mu (s/T)) \exp (-i\omega_l (t-s)) \right]
$$

$$
+ \sum_{j_1 \neq j_2} \frac{1}{T^2} \mathbb{E} \left[ \sum_{t=\lfloor T \lambda_{j_1-1}^0 \rfloor +1}^{\lfloor T \lambda_{j_1}^0 \rfloor} \sum_{s=\lfloor T \lambda_{j_2-1}^0 \rfloor +1}^{\lfloor T \lambda_{j_2}^0 \rfloor} (X_t - \mu (t/T)) (X_s - \mu (s/T)) \exp (-i\omega_l (t-s)) \right]
$$

$$
+ \left[ \frac{1}{\sqrt{T}} \frac{\exp(-i\omega_l)}{1 - \exp(-i\omega_l)} \sum_{j=1}^{m_0+1} B_j \left( \exp \left( -i\omega \lfloor T \lambda_{j-1}^0 \rfloor \right) \right) - \exp \left( -i\omega \lfloor T \lambda_j^0 \rfloor \right) \right]^2 + o(1)
$$

$$
= \sum_{j=1}^{m_0+1} \left( \frac{1}{T} \sum_{t=\lfloor T \lambda_{j-1}^0 \rfloor +1}^{\lfloor T \lambda_j^0 \rfloor} (X_t - \mu (t/T))^2 + \frac{2}{T r_j} \sum_{k=1}^{\lfloor T r_j \rfloor -1} \sum_{t=\lfloor T \lambda_{j-1}^0 \rfloor +k+1}^{\lfloor T \lambda_j^0 \rfloor} \Gamma_{t/T} (k) \exp (-i\omega_l k) \right)
$$

$$
+ \left[ \frac{1}{\sqrt{T}} \frac{\exp(-i\omega_l)}{1 - \exp(-i\omega_l)} \sum_{j=1}^{m_0+1} B_j \left( \exp \left( -i\omega \lfloor T \lambda_{j-1}^0 \rfloor \right) \right) - \exp \left( -i\omega \lfloor T \lambda_j^0 \rfloor \right) \right]^2 + o(1).
$$

Next, using the definition of $f(u, \omega_l)$, $e^{-2i\omega_l} = 1$ by Euler’s formula and letting $\omega_l \to 0$ we have,

$$
\mathbb{E} \left( I_T (\omega_l) \right) \geq \sum_{j=1}^{m_0+1} \left( \int_{\lfloor T \lambda_{j-1}^0 \rfloor}^{\lfloor T \lambda_j^0 \rfloor} c(u, 0) du + 2 \sum_{k=1}^{\infty} \int_{\lfloor T \lambda_{j-1}^0 \rfloor}^{\lfloor T \lambda_j^0 \rfloor} c(u, k) \exp (-i\omega_l k) du \right)
$$

$$
+ \frac{1}{T} \frac{1}{1 - \exp(-i\omega_l)} \left[ B_1 - B_{m_0+1} - (1 + o(1)) \sum_{j=1}^{m_0} (B_j - B_{j+1}) \exp \left( -2\pi i l \lambda_j^0 \right) \right]^2 + o(1)
$$

$$
= 2\pi \sum_{j=1}^{m_0+1} r_j \int_{\lfloor T \lambda_{j-1}^0 \rfloor}^{\lfloor T \lambda_j^0 \rfloor} f(u, \omega_l) du
$$

$$
+ \frac{1}{T} \frac{1}{1 - \exp(-i\omega_l)} \left[ B_1 - B_{m_0+1} - (1 + o(1)) \sum_{j=1}^{m_0} (B_j - B_{j+1}) \exp \left( -2\pi i l \lambda_j^0 \right) \right]^2 + o(1)
$$

$$
= 2\pi \int_0^1 f(u, \omega_l) du + \frac{1}{T \omega_l^2} \left[ B_1 - B_{m_0+1} - \sum_{j=1}^{m_0} (B_j - B_{j+1}) \exp \left( -2\pi i l \lambda_j^0 \right) \right]^2 + o(1).
$$

(7.7)

By Assumption 2.1-(ii), the first term of (7.7) is bounded for all frequencies $\omega_j$. Since $B_1, \ldots, B_{m_0+1}$ are arbitrary, if $T \omega_l^2 \to 0$ then the order of the second term of (7.7) is $O((T \omega_l^2)^{-1})$. Note that as $\omega_l \to 0$ there
are some values of \(l\) for which the corresponding term involving \(|·|^2\) on the right-hand side of (7.7) is equal to zero [see the argument in Mikosch and Stáricá (2004)]. In such a case, \(\mathbb{E}(I_T(\omega)) \geq 2\pi \int_0^1 f(u, \omega) \, du > 0\).

For the other values of \(\{l\}\) as \(\omega_l \to 0\), the second term of (7.7) diverges to infinity. The outcome is that there are frequencies close to \(\omega_l = 0\) for which \(\mathbb{E}(I_T(\omega)) \to \infty\). \(\square\)

### 7.3 Proof of Theorem 3.3

We consider the case \(k \geq 0\). The case \(k < 0\) follows similarly. Consider any \(u \in (0, 1)\) such that \(T_j^0 \notin [Tu + k/2 - n_{2,T}/2 + 1, \ldots, Tu + n_{2,T}/2]\) for all \(j = 1, \ldots, m_0\). Theorem 3.3 in Casini and Perron (2021a) shows that

\[
\mathbb{E} [\tilde{c}_T(u, k)] = c(u, k) + \frac{1}{2} \frac{n_{2,T}}{T^2} \left( \frac{\partial^2}{\partial u^2} c(u, k) \right) + o \left( \frac{n_{2,T}}{T^2} \right) + O \left( \frac{1}{n_{2,T}} \right). \tag{7.8}
\]

Since \(n_{2,T} \to \infty\) and \(n_{2,T}/T \to 0\), \(\mathbb{E} [\tilde{c}_T(u, k)] = c(u, k) + o(1)\). The same aforementioned theorem shows that \(n_{2,T} \mathbb{E} \left[ \tilde{c}_T(u, k) \right] = O_T(1)\). This combined with (7.8) yields part (i) of the theorem.

Next, we consider case (ii-a) with \((T_j^0 - (Tu + k/2 - n_{2,T}/2 + 1))/n_{2,T} \to \gamma \in (0, 1)\). We have,

\[
\tilde{c}_T(u, k) = n_{2,T}^{-1} \sum_{s=0}^{n_{2,T}} X_{[Tu]} + k/2 - n_{2,T}/2 + s + 1 \cdot X_{[Tu]} + k/2 - n_{2,T}/2 + s + 1 - k \left( n_{2,T}^{-1} \sum_{s=0}^{n_{2,T}} X_{[Tu]} + k/2 - n_{2,T}/2 + s + 1 \right)^2
\]

\[
= n_{2,T}^{-1} \sum_{s=0}^{n_{2,T}} X_{[Tu]} + k/2 - n_{2,T}/2 + s + 1 - k - \mathbb{E} \left( X_{[Tu]} + k/2 - n_{2,T}/2 + s + 1 \right) \mathbb{E} \left( X_{[Tu]} + k/2 - n_{2,T}/2 + s + 1 - k \right)
\]

\[
+ n_{2,T}^{-1} \sum_{s=0}^{n_{2,T}} X_{[Tu]} + k/2 - n_{2,T}/2 + s + 1 - k - \mathbb{E} \left( X_{[Tu]} + k/2 - n_{2,T}/2 + s + 1 \right) \mathbb{E} \left( X_{[Tu]} + k/2 - n_{2,T}/2 + s + 1 - k \right)
\]

\[
+ n_{2,T}^{-1} \sum_{s=0}^{n_{2,T}} X_{[Tu]} + k/2 - n_{2,T}/2 + s + 1 - k - \mathbb{E} \left( X_{[Tu]} + k/2 - n_{2,T}/2 + s + 1 \right) \mathbb{E} \left( X_{[Tu]} + k/2 - n_{2,T}/2 + s + 1 - k \right)
\]

\[
+ n_{2,T}^{-1} \sum_{s=0}^{n_{2,T}} X_{[Tu]} + k/2 - n_{2,T}/2 + s + 1 - k - \mathbb{E} \left( X_{[Tu]} + k/2 - n_{2,T}/2 + s + 1 \right) \mathbb{E} \left( X_{[Tu]} + k/2 - n_{2,T}/2 + s + 1 - k \right)
\]

\[
+ n_{2,T}^{-1} \sum_{s=0}^{n_{2,T}} X_{[Tu]} + k/2 - n_{2,T}/2 + s + 1 - k - \mathbb{E} \left( X_{[Tu]} + k/2 - n_{2,T}/2 + s + 1 \right) \mathbb{E} \left( X_{[Tu]} + k/2 - n_{2,T}/2 + s + 1 - k \right)
\]

\[
+ n_{2,T}^{-1} \sum_{s=0}^{n_{2,T}} X_{[Tu]} + k/2 - n_{2,T}/2 + s + 1 - k - \mathbb{E} \left( X_{[Tu]} + k/2 - n_{2,T}/2 + s + 1 \right) \mathbb{E} \left( X_{[Tu]} + k/2 - n_{2,T}/2 + s + 1 - k \right)
\]
Theorem 3.3 in Casini and Perron (2021a) shows that
\[ u \in C^2 \]

Consider first any \( d \) \( \in \mathbb{R} \) and \( \nu \geq 0 \). Consider the case (ii-b) with \( T \)
\[ T_0 = (T + k/2 - n_{2,T}/2 + 1) \]
\[ n_{2,T} = \sum_{s=0}^{n_{2,T}} X_{[Tu] + k/2 - n_{2,T}/2 + s + 1} \]
\[ \geq \gamma c \left( \lambda^0_j, k \right) + (1 - \gamma) c (u, k) + \gamma \mu_j \left( \lambda^0_j \right)^2 + (1 - \gamma) \mu_{j+1} (u)^2 
- \gamma \mu_j \left( \lambda^0_j \right) + (1 - \gamma) \mu_{j+1} (u)^2 + o_T (1) 
= \gamma c \left( \lambda^0_j, k \right) + (1 - \gamma) c (u, k) + \gamma (1 - \gamma) \left( \mu_j \left( \lambda^0_j \right) - \mu_{j+1} (u)^2 \right) + o_T (1) . \]

Consider the case (ii-b) with \( (T_0 - (Tu + k/2 - n_{2,T}/2 + 1))/n_{2,T} \to 0 \). The other case follows by symmetry. Eq. (7.9) continues to hold. The first term, third term and the first summation of the last term on the right-hand side of (7.9) are negligible. Thus, using ergodicity, implied by Assumption 2.1-2.2-(i),
\[ \hat{c}_T (u, k) = c (u, k) + n_{2,T}^{-1} \sum_{s=T_0}^{n_{2,T}} \mathbb{E} \left( X_{[Tu] + k/2 - n_{2,T}/2 + s + 1} \right) \mathbb{E} \left( X_{[Tu] + k/2 - n_{2,T}/2 + s + 1} \right) 
- \mu (u)^2 + o_T (1) 
= c (u, k) + \mu_{j+1} (u)^2 - \mu_{j+1} (u)^2 + o_T (1) = c (u, k) + o_T (1) , \]
where we have used the smoothness of \( \mathbb{E} (X_t) \) implied by local stationarity. The second claim of the lemma follows from Assumption 2.2-(i) since this implies that \( \sup_{u \in [0, 1]} c (u, k) \to 0 \) as \( k \to \infty \) and the fact that the third term on the right-hand side of (7.10) does not depend on \( k \). Thus, \( \hat{\Gamma}_D (k) \geq d_T^c + o_T (1) \) where
\[ d_T^c = (n_{T} / T) \gamma (1 - \gamma) \left( \mu_j \left( \lambda^0_j \right) - \mu_{j+1} (u)^2 > 0 \right) \text{ and } d_T^c \to 0 \text{ since } n_{T} / T \to 0 . \]

### 7.4 Proof of Theorem 3.4

Consider first any \( u \in (0, 1) \) such that \( T_j^0 \notin \{ Tu - n_{T}/2 + 1, \ldots, Tu + n_{T}/2 \} \) for all \( j = 1, \ldots, m_0 \).

Theorem 3.3 in Casini and Perron (2021a) shows that
\[ \mathbb{E} (I_{L,T} (u, \omega)) = \left| \frac{1}{\sqrt{n_T}} \sum_{s=0}^{n_{T}-1} V_{[Tu] - n_{T}/2 + s + 1, T} \exp (-i \omega \xi s) \right|^2 \]
\[ = f (u, \omega) + \frac{1}{6} \left( \frac{n_{T}}{T} \right)^2 \left( \frac{\partial^2}{\partial u^2} f (u, \omega) \right) + o \left( \left( \frac{n_{T}}{T} \right)^2 \right) \]

By Assumption 2.1 the absolute value of the first term on the right-hand side is bounded for all frequencies \( \omega \). By Assumption 3.2-(iii) \( \left| \left( \frac{\partial^2}{\partial u^2} f (u, \omega) \right) \right| \) is bounded and, since \( n_{T}/T \to 0 \), the second term converges to zero. Similarly, the third and fourth terms are negligible. Thus, \( \mathbb{E} (I_{L,T} (u, \omega)) \) is bounded below by \( f (u, \omega) \) as \( \omega \to 0 \) which establishes part (i). Now we consider to part (ii). We begin with case (ii). We only focus on the sub-case \( (T_0 - (Tu - n_{T}/2 + 1))/n_{T} \to \gamma \) with \( \gamma \in (0, 1) \). We have
\[ I_{L,T} (\omega) = \left| \frac{1}{\sqrt{n_T}} \sum_{s=0}^{n_{T}-1} X_{[Tu] - n_{T}/2 + s + 1, T} \exp (-i \omega \xi s) + \sum_{s=T_0}^{n_{T}-1} X_{[Tu] - n_{T}/2 + s + 1, T} \exp (-i \omega \xi s) \right|^2 \]
\[
\begin{align*}
&= \frac{1}{n_T} \left[ T_j^0 - ([Tu] - n_T/2 + 1) \sum_{s=0}^{n_T-1} (X_{[Tu]-n_T/2+s+1,T} - \mu \left( ([Tu] - n_T/2 + s + 1)/T \right)) \exp (-i\omega_l s) \right. \\
&\quad + \sum_{s=T_j^0 - ([Tu]-n_T/2)}^{n_T-1} (X_{[Tu]-n_T/2+s+1,T} - \mu \left( ([Tu] - n_T/2 + s + 1)/T \right)) \exp (-i\omega_l s) \\
&\quad + \sum_{s=0}^{n_T-1} \mu \left( ([Tu] - n_T/2 + s + 1)/T \right) \exp (-i\omega_l s) \bigg] .
\end{align*}
\]

Using Assumption 3.2, we have
\[
\begin{align*}
&\left| \sum_{s=0}^{n_T-1} \mu \left( ([Tu] - n_T/2 + s + 1)/T \right) \exp (-i\omega_l s) \right|^2 \\
&\geq \left| T_j^0 - ([Tu] - n_T/2 + 1) \sum_{s=0}^{n_T-1} \exp (-i\omega_l s) + B_{j+1} \sum_{s=T_j^0 - ([Tu]-n_T/2)}^{n_T-1} \exp (-i\omega_l s) \right|^2 .
\end{align*}
\]

Note that
\[
\begin{align*}
&= B_j \sum_{s=0}^{n_T-1} \exp (-i\omega_l s) \\
&\quad + B_{j+1} \exp \left( -i\omega_l \left( T_j^0 - ([Tu] - n_T/2) \right) \right) \sum_{s=0}^{n_T-1} \exp (-i\omega_l s) .
\end{align*}
\]

Focusing on the second term on the right-hand side above,
\[
\begin{align*}
&n_T^{-1} \left| B_{j+1} \sum_{s=T_j^0 - ([Tu]-n_T/2)}^{n_T-1} \exp (-i\omega_l s) \right|^2 \\
&= n_T^{-1} \left| B_{j+1} \exp \left( -i\omega_l \left( T_j^0 - ([Tu] - n_T/2) \right) \right) \sum_{s=0}^{n_T-1} \exp (-i\omega_l s) \right|^2 \\
&= n_T^{-1} \left| B_{j+1} \exp \left( -i\omega_l \left( T_j^0 - ([Tu] - n_T/2) \right) \right) \frac{1 - \exp \left( -i\omega_l \left( n_T - \left( T_j^0 - ([Tu] - n_T/2) \right) \right) \right)}{1 - \exp (-i\omega_l)} \right|^2 \\
&= n_T^{-1} \left| B_{j+1} \frac{\exp \left( -i\omega_l \left( T_j^0 - ([Tu] - n_T/2) \right) \right) - \exp (-i\omega_l n_T)}{1 - \exp (-i\omega_l)} \right|^2 .
\end{align*}
\]

We show that the above equation diverges to infinity as \( \omega_l \to 0 \) with \( n_T \omega_l^2 \to 0 \). If \( n_T \omega_l \to a \in (0, \infty) \) then \( \text{Re} \left( \exp (-i\omega_l n_T) \right) \neq 1 \) and the order is determined by the denominator. As in the proof of Theorem 3.2, \( |1 - \exp(-i\omega_l)|^2 = \omega_l^2 \). Since \( n_T \omega_l^2 \to 0 \), the right-hand side above diverges. If \( n_T \omega_l \to 0 \), we apply
L'Hôpital's rule to obtain

\[
\begin{align*}
&n_T^{-1} \left| B_{j+1} \frac{-i \left( T_0^j - ([Tu] - n_T/2) \right) + in_T}{i} \right|^2 \\
&= n_T^{-1} B_{j+1}^2 \left( - \left( T_0^j - ([Tu] - n_T/2) \right)^2 + n_T^2 - \left( T_0^j - ([Tu] - n_T/2) \right) n_T \right) \\
&= O \left( n_T^2 / n_T \right) = O \left( n_T \right),
\end{align*}
\]

which shows that the right-hand side of (7.15) diverges. A similar argument can be applied to the first term on the right-hand side of (7.14) and to the product of the latter term and the complex conjugate of the second term on the right-hand side of (7.15).

It remains to consider case (b) and the sub-case \( T_0^j - (Tu - n_T/2 + 1) / n_T \to 0 \). The other sub-case follows by symmetry. We have (7.12) and (7.13). Note that,

\[
\begin{align*}
&\left| \frac{1}{\sqrt{n_T}} B_{j+1} \sum_{s=T_0^j - ([Tu] - n_T/2)}^{n_T-1} \exp \left( -i \omega_l s \right) \right|^2 \\
&= \left| \frac{1}{\sqrt{n_T}} B_{j+1} \sum_{s=0}^{n_T-1} \exp \left( -i \omega_l s \right) - \frac{1}{\sqrt{n_T}} B_{j+1} \sum_{s=0}^{T_0^j - ([Tu] - n_T/2) - 1} \exp \left( -i \omega_l s \right) \right|^2 \\
&= \left| - \frac{1}{\sqrt{n_T}} B_{j+1} \sum_{s=0}^{T_0^j - ([Tu] - n_T/2) - 1} \exp \left( -i \omega_l s \right) \right|^2 \to 0.
\end{align*}
\]

Thus, we have

\[
\mathbb{E} (I_{LT} (\omega_l)) = \frac{1}{n_T} \left( \sum_{s=0}^{T_0^j - ([Tu] - n_T/2) + s + 1, T} \left( X_{[Tu] - n_T/2 + s + 1, T} - \mu \left( ([Tu] - n_T/2 + s + 1) / T \right) \right) \exp \left( -i \omega_l s \right) \right) \\
+ \sum_{s=T_0^j - ([Tu] - n_T/2)}^{n_T-1} \left( X_{[Tu] - n_T/2 + s + 1, T} - \mu \left( ([Tu] - n_T/2 + s + 1) / T \right) \right) \exp \left( -i \omega_l s \right)^2 + O(1).
\]

Note that the first sum above involves at most \( C < \infty \) summands. So the first term is negligible. The expectation of the product of the first term and the conjugate of the second term is negligible by using arguments similar to the proof in Lemma 7.1 with \( n_T \) in place of \( T \). Thus, the limit of \( \mathbb{E} (I_{LT} (\omega_l)) \) is equal to the right-hand side of (7.11) plus additional \( o(1) \) terms. □