Supplement to Continuous Record Laplace-based Inference about the Break Date in Structural Change Models

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Abstract

This supplemental material includes the Mathematical Appendix to Casini and Perron (2020b), which includes all proofs of the results in the paper.

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A Mathematical Appendix

The supplement is structured as follows. In Section A.1 we present some preliminary lemmas. The proof of the results of Section 4 can be found in Section A.2 while those about inference are in Section A.3.

A.1 Additional Notation and Preliminary Results

We denote the $(i,\, j)$-th element of the outer product matrix $A'A$ by $(A'A)_{i,j}$ and the $i\times j$ upper-left (resp., lower-right) sub-block of $A'A$ by $[A'A]_{\{i\times j,\cdot\}}$ (resp., $[A]_{\{\cdot,i\times j\}}$). For a random variable ξ and a number $r\geq 1$, we write $\|\xi\|_r=(\mathbb{E}\,\|\xi\|^r)^{1/r}$. C is used as a generic constant that may vary from line to line; we may sometime write C_r to emphasize the dependence of C on some number r. For two scalars a and b the symbol $a\wedge b$ means the infimum of $\{a,\, b\}$. The symbol "$\overset{\text{u.c.p.}}{\Rightarrow}$" signifies uniform locally in time convergence under the Skorokhod topology, which implies convergence in probability.

It is typical in the high frequency statistics literature to use a localization argument [cf. Section I.1.d in Jacod and Shiryaev (2003)] which allows us to replace Assumption 2.1-2.2 by the following stronger assumption which basically turns all the local conditions into global ones.

Assumption A.1. *Under Assumption 2.1-2.6, $\{Y_t,\, D_t,\, Z_t\}_{t\geq 0}$ takes value in some compact set, $\{\sigma_{\cdot,t},\, \sigma_{e,t}\}_{t\geq 0}$ is bounded càdlàg and $\{\mu_{\cdot,t}\}$ are bounded càdlàg or càglàd.*

Next, we collect a few lemmas. They follow from Lemma S.A.1-S.A.4 in CP (2020a).

Lemma A.1. *The following inequalities hold $\mathbb{P}$-a.s.:*

$$(Z_0'MZ_0)-(Z_0'MZ_2)\,(Z_2'MZ_2)^{-1}\,(Z_2'MZ_0)\geq H'\,(X_\Delta'X_\Delta)\,(X_2'X_2)^{-1}\,(X_0'X_0)\,H,\qquad T_b<T_b^0 \tag{A.1}$$

$$(Z_0'MZ_0)-(Z_0'MZ_2)\,(Z_2'MZ_2)^{-1}\,(Z_2'MZ_0)\geq H'\,(X_\Delta'X_\Delta)\,(X'X-X_2'X_2)^{-1}\,(X'X-X_0'X_0)\,H,\quad T_b\geq T_b^0 \tag{A.2}$$

Lemma A.2. *Let X_t $\left(\widetilde{X}_t\right)$ be a q-dimensional (resp. p-dimensional) Itô continuous semimartingale defined on $[0,\, N]$. Let Σ_t denote the date t instantaneous covariation between X_t and $\widetilde{X}_t$. Choose a fixed number $\epsilon>0$ and ϖ satisfying $1/2-\epsilon\geq\varpi\geq\epsilon>0$. Further, let $B_T\triangleq\lfloor N/h-T^{\varpi}\rfloor$. Define the moving average of Σ_t as $\overline{\Sigma}_{kh}\triangleq(T^{\varpi}h)^{-1}\int_{kh}^{kh+T^{\varpi}h}\Sigma_s ds$, and let $\widehat{\Sigma}_{kh}\triangleq(T^{\varpi}h)^{-1}\sum_{i=1}^{\lfloor T^{\varpi}\rfloor}\Delta_h X_{k+i}\Delta_h\widetilde{X}_{k+i}'$. Then, $\sup_{1\leq k\leq B_T}||\widehat{\Sigma}_{kh}-\overline{\Sigma}_{kh}||=o_{\mathbb{P}}(1)$.*

Lemma A.3. *Under Assumption A.1 we have as $h\downarrow 0$, $T\rightarrow\infty$ with N fixed and for any $1\leq i,\, j\leq p$, (i) $\left|(Z_2'e)_{i,1}=\sum_{k=T_b+1}^{T}z_{kh}^{(i)}e_{kh}\right|\overset{\mathbb{P}}{\rightarrow}0$; (ii) $\left|(Z_0'e)_{i,1}=\sum_{k=T_b^0+1}^{T}z_{kh}^{(i)}e_{kh}\right|\overset{\mathbb{P}}{\rightarrow}0$; (iii) $\left|(Z_2'Z_2)_{i,j}-\int_{(T_b+1)h}^{N}\Sigma_{ZZ,s}^{(i,j)}ds\right|\overset{\mathbb{P}}{\rightarrow}0$ where $(Z_2'Z_2)_{i,j}=\sum_{k=T_b+1}^{T}z_{kh}^{(i)}z_{kh}^{(j)}$; (iv) $\left|(Z_0'Z_0)_{i,j}-\int_{(T_b^0+1)h}^{N}\Sigma_{ZZ,s}^{(i,j)}ds\overset{\mathbb{P}}{\rightarrow}\right|0$ with $(Z_0'Z_0)_{i,j}=\sum_{k=T_b^0+1}^{T}z_{kh}^{(i)}z_{kh}^{(j)}$. For the following estimates involving X, we have, for any $1\leq r\leq p$ and $1\leq l\leq q+p$, (v) $\left|(Xe)_{l,1}\right|\overset{\mathbb{P}}{\rightarrow}0$ where $(Xe)_{l,1}=\sum_{k=1}^{T}x_{kh}^{(l)}e_{kh}$; (vi) $\left|(Z_2'X)_{r,l}-\int_{(T_b+1)h}^{N}\Sigma_{ZX,s}^{(r,l)}ds\right|\overset{\mathbb{P}}{\rightarrow}0$ where $(Z_2'X)_{r,l}=\sum_{k=T_b+1}^{T}z_{kh}^{(r)}x_{kh}^{(l)}$; (vii) $\left|(Z_0'X)_{r,l}-\int_{(T_b^0+1)h}^{N}\Sigma_{ZX,s}^{(r,l)}ds\right|\overset{\mathbb{P}}{\rightarrow}0$ where $(Z_0'X)_{r,l}=\sum_{k=T_b^0+1}^{T}z_{kh}^{(r)}x_{kh}^{(l)}$. Further, for $1\leq u,\, d\leq q+p$, (viii) $\left|(X'X)_{u,d}-\int_0^N\Sigma_{XX,s}^{(u,d)}ds\right|\overset{\mathbb{P}}{\rightarrow}0$ where $(X'X)_{u,d}=\sum_{k=1}^{T}x_{kh}^{(u)}x_{kh}^{(d)}$.*

Denote

$$X_\Delta\triangleq X_2-X_0=\left(0,\,\ldots,\,0,\,x_{(T_b+1)h},\ldots,\,x_{T_b^0h},\,0,\ldots,\right)',\qquad\text{for } N_b<N_b^0$$

$$X_\Delta\triangleq-(X_2-X_0)=\left(0,\,\ldots,\,0,\,x_{(T_b^0+1)h},\ldots,\,x_{T_bh},\,0,\ldots,\right)',\qquad\text{for } N_b>N_b^0$$

whereas $X_\Delta \triangleq 0$ for $N_b = N_b^0$. Observe that $X_2 = X_0 + X_\Delta \text{sign}\left(N_b^0 - N_b\right)$. When the sign is immaterial, we simply write $X_2 = X_0 + X_\Delta$.

Lemma A.4. *Under Assumption A.1, we have as $h \downarrow 0$, $T \to \infty$ with N fixed, $|N_b^0 - N_b| > \gamma > 0$ and for any $1 \leq i,\, j \leq p$,*

(i) with $(Z'_\Delta Z_\Delta)_{i,j} = \sum_{k=T_b^0+1}^{T_b} z_{kh}^{(i)} z_{kh}^{(j)}$ *then* $\begin{cases} |\,(Z'_\Delta Z_\Delta)_{i,j} - \int_{(T_b+1)h}^{T_b^0 h} \Sigma_{ZZ,s}^{(i,j)} ds| \xrightarrow{\mathbb{P}} 0, & \text{if } T_b < T_b^0 \\ |\,(Z'_\Delta Z_\Delta)_{i,j} - \int_{T_b^0 h}^{(T_b+1)h} \Sigma_{ZZ,s}^{(i,j)} ds| \xrightarrow{\mathbb{P}} 0, & \text{if } T_b > T_b^0 \end{cases};$

and for $1 \leq r \leq p + q$

(ii) with $(Z'_\Delta X_\Delta)_{i,r} = \sum_{k=T_b^0+1}^{T_b} z_{kh}^{(i)} x_{kh}^{(r)}$ *then* $\begin{cases} |\,(Z'_\Delta X_\Delta)_{i,r} - \int_{(T_b+1)h}^{T_b^0 h} \Sigma_{ZX,s}^{(i,r)} ds| \xrightarrow{P} 0, & \text{if } T_b < T_b^0 \\ |\,(Z'_\Delta X_\Delta)_{i,r} - \int_{T_b^0 h}^{(T_b+1)h} \Sigma_{ZX,s}^{(i,r)} ds| \xrightarrow{P} 0, & \text{if } T_b > T_b^0 \end{cases}.$

Moreover, each quantity on the right hand side above are bounded in probability.

We will also use the following Central Limit Theorem [see Theorem 5.4.2. in Jacod and Protter (2012)]. We choose a progressively measurable "square-root" process Υ_Z of the $\mathbb{M}_{q^2}^+$-valued process $\widehat{\Sigma}_{Z,s}$ whose elements are given by $\widehat{\Sigma}_{Z,s}^{(ij,kl)} = \Sigma_{Z,s}^{(ik)} \Sigma_{Z,s}^{(jl)}$. Note that by the symmetry of $\Sigma_{Z,s}$, the matrix with entries $(\Upsilon_{Z,s}^{(ij,kl)} + \Upsilon_{Z,s}^{(ji,kl)})/\sqrt{2}$ is a square-root of the matrix with entries $\widehat{\Sigma}_{Z,s}^{(ij,kl)} + \widehat{\Sigma}_{Z,s}^{(il,jk)}$. Let W_s^* a q^2-dimensional standard Wiener process defined on an extension of $(\Omega,\, \mathscr{F},\, \{\mathscr{F}_t\},\, \mathbb{P})$. Then the process $\mathcal{U}_t$ with components $\mathcal{U}_t^{(rj)} = 2^{-1/2} \sum_{k,l=1}^q \int_0^t (\Upsilon_{Z,s}^{(rj,kl)} + \Upsilon_{Z,s}^{(jr,kl)}) dW_s^{*(kl)}$ is, conditionally on $\mathscr{F}$, a continuous Gaussian process with independent increments and (conditional) covariance $\widetilde{\mathbb{E}}(\mathcal{U}^{(rj)}(v)\,\mathcal{U}^{(kl)}(v)\,|\,\mathscr{F}) = \int_{N_b^0+v}^{N_b^0} (\Sigma_{Z,s}^{(rk)} \Sigma_{Z,s}^{(jl)} + \Sigma_{Z,s}^{(rl)} \Sigma_{Z,s}^{(jk)}) ds$, with $v < 0$. The Central Limit Theorem we shall use is the following [cf. Lemma S.A.5 in CP (2020a)].

Lemma A.5. *Let Z be a continuous Itô semimartingale satisfying Assumption A.1. Then,* $h^{-1/2}(Z'_2 Z_2 - ([Z,\, Z]_{Th} - [Z,\, Z]_{(T_b+1)h})) \overset{\mathcal{L}-s}{\Rightarrow} \mathcal{U}$.

Recall $Q_h(\widehat{\delta}(N_b),\, N_b) = \widehat{\delta}(N_b)'(Z'_2 M Z_2)\widehat{\delta}(N_b)$. We decompose $Q_h(\widehat{\delta}(N_b),\, N_b) - Q_h(\widehat{\delta}\left(N_b^0\right),\, N_b^0)$ into a "deterministic" and a "stochastic" component. By definition,

$$\widehat{\delta}\left(N_b\right) = \left(Z'_2 M Z_2\right)^{-1} \left(Z'_2 M Y\right) = \left(Z'_2 M Z_2\right)^{-1} \left(Z'_2 M Z_0\right) \delta^0 + \left(Z'_2 M Z_2\right)^{-1} Z_2 M e,$$

and $\widehat{\delta}\left(N_b^0\right) = \left(Z'_0 M Z_0\right)^{-1} \left(Z'_0 M Y\right) = \delta^0 + \left(Z'_0 M Z_0\right)^{-1} \left(Z'_0 M e\right)$. Therefore,

$$\begin{aligned} Q_h\left(\widehat{\delta}\left(N_b\right),\, N_b\right) - Q_h\left(\widehat{\delta}\left(N_b^0\right),\, N_b^0\right) &= \widehat{\delta}\left(N_b\right)\left(Z'_2 M Z_2\right)\widehat{\delta}\left(N_b\right) - \widehat{\delta}\left(N_b^0\right)'\left(Z'_0 M Z_0\right)\widehat{\delta}\left(N_b^0\right) \\ &\triangleq D_h\left(\delta^0,\, N_b\right) + S_{e,h}\left(\delta^0,\, N_b\right), \end{aligned} \tag{A.3}$$

where

$$D_h\left(\delta^0,\, N_b\right) \triangleq \left(\delta^0\right)'\left\{\left(Z'_0 M Z_2\right)\left(Z'_2 M Z_2\right)^{-1}\left(Z'_2 M Z_0\right) - Z'_0 M Z_0\right\}\delta^0 \tag{A.4}$$

is the deterministic part and

$$S_{e,h}\left(\delta^0,\, N_b\right) \triangleq 2\left(\delta^0\right)'\left(Z'_0 M Z_2\right)\left(Z'_2 M Z_2\right)^{-1} Z_2 M e - 2\left(\delta^0\right)'\left(Z'_0 M e\right) \tag{A.5}$$

$$+ e' M Z_2 \left(Z'_2 M Z_2\right)^{-1} Z_2 M e - e' M Z_0 \left(Z'_0 M Z_0\right)^{-1} Z'_0 M e \tag{A.6}$$

is the stochastic part. Next, let $Z_\Delta = X_\Delta H$, and define

$$\overline{D}_h\left(\delta^0,\, N_b\right) \triangleq -D_h\left(\delta^0,\, N_b\right) / \left|T_b - T_b^0\right|. \tag{A.7}$$

We arbitrarily define $\overline{D}_h(\delta^0, N_b) = (\delta^0)' \delta^0$ when $N_b = N_b^0$ since both the numerator and denominator of $\overline{D}_h(\delta^0, N_b)$ are zero. Observe that $\overline{D}_h(\delta^0, N_b)$ is non-negative because the matrix inside the braces in $D_h(\delta^0, N_b)$ is negative semidefinite. Equation (A.3) can be rewritten as

$$Q_h(\delta(N_b), N_b) - Q_h\left(\delta\left(N_b^0\right), N_b^0\right) = -\left|T_b - T_b^0\right| \overline{D}_h\left(N_b, \delta^0\right) + S_{e,h}\left(N_b, \delta^0\right) \qquad \text{for all } T_b. \tag{A.8}$$

Lemma A.6. *Under Assumption 2.2-2.6 and 3.1, uniformly in N_b,*

$$Q_h(\delta(N_b), N_b) - Q_h\left(\delta\left(N_b^0\right), N_b^0\right) = -\delta_h(Z'_\Delta Z_\Delta)\delta_h + 2\delta'_h(Z'_\Delta \widetilde{e})\operatorname{sgn}\left(T_b^0 - T_b\right) + o_{\mathbb{P}}\left(h^{3/2-\kappa}\right). \tag{A.9}$$

Proof. It follows from Lemma 4.1 in CP (2020a). □

Lemma A.7. *For any $\epsilon > 0$ there exists a $C > 0$ such that*

$$\liminf_{h\downarrow 0} \mathbb{P}\left[\sup_{K\leq|u|\leq\eta T^{1-\kappa}} Q_h\left(N_b^0 + u/r_h\right) - Q_h\left(N_b^0\right) < -C\right] \geq 1-\epsilon,$$

for some large K and small $\eta > 0$.

Proof. Note that on $\{K \leq |u| \leq \eta T^{1-\kappa}\}$ we have $KT^\kappa \leq |T_b - T_b^0| < \eta T$. Suppose $N_b < N_b^0$. Let $\mathbf{T}_{K,\eta} = \{T_b : N\eta > |N_b - N_b^0| > K(T^{1-\kappa})^{-1}\}$. It is enough to show that

$$\mathbb{P}\left(\sup_{T_b\in\mathbf{T}_{K,\eta}} Q_h\left(N_b^0 + u/r_h\right) - Q_h\left(N_b^0\right) \geq 0\right) < \epsilon,$$

or using (A.8), $\mathbb{P}\left(\sup_{T_b\in\mathbf{T}_{K,\eta}} h^{-3/2} S_{e,h}(\delta_h, N_b) / |T_b - T_b^0| \geq \inf_{T_b\in\mathbf{T}_{K,\eta}} h^{-3/2}\overline{D}_h(\delta_h, N_b)\right) < \epsilon$. The difficulty is to control the estimates that depend on the difference $|N_b - N_b^0|$. By Lemma A.1,

$$\inf_{T_b\in\mathbf{T}_{K,\eta}} \overline{D}_h(\delta_h, N_b) \geq \inf_{T_b\in\mathbf{T}_{K,\eta}} \delta'_h H'\left(X'_\Delta X_\Delta / \left|T_b - T_b^0\right|\right)(X'_2X_2)^{-1}(X'_0X_0)H\delta_h,$$

and, since $T\eta > |T_b - T_b^0| \geq KT^\kappa/N$, we need to study the behavior of $X'_\Delta X_\Delta = \sum_{k=T_b+1}^{T_b^0} x_{kh}x'_{kh}$. We shall apply asymptotic results for the local approximation of the covariation between processes. By Theorem 9.3.2 part (i) in Jacod and Protter (2012),

$$\left(h\left(T_b^0 - T_b\right)\right)^{-1} \sum_{k=T_b+1}^{T_b^0} x_{kh}x'_{kh} \xrightarrow{\mathbb{P}} \Sigma_{XX,N_b^0}, \tag{A.10}$$

since $|N_b - N_b^0|$ shrinks at a rate no faster than $h^{1-\kappa}$ and $(1/h^{1-\kappa}) \to \infty$. Since X'_0X_0 and X'_2X_2 both involve at least a fixed positive fraction of the data, we can apply a simple law of large numbers for approximate covariation processes [cf. Lemma A.3] to show that X'_0X_0 and X'_2X_2 are each $O_{\mathbb{P}}(1)$. By Lemma A.2, the approximation in (A.10) is uniform, and thus

$$\begin{aligned} h^{-1/2} \inf_{T_b\in\mathbf{T}_{K,\eta}} &\delta'_h H'\left(X'_\Delta X_\Delta / h\left|T_b - T_b^0\right|\right)(X'_2X_2)^{-1}(X'_0X_0)H\delta_h \\ &= \inf_{T_b\in\mathbf{T}_{K,\eta}} \left(\delta^0\right)' H'\left(X'_\Delta X_\Delta / h\left|T_b - T_b^0\right|\right)(X'_2X_2)^{-1}(X'_0X_0)H\delta^0 \end{aligned}$$

is bounded away from zero, in view of Assumption 2.2-(iii). It remains to show that

$$\mathbb{P}\left(\sup_{T_b\in\mathbf{T}_{K,\eta}} h^{-3/2} S_{e,h}(\delta_h, N_b) / \left(T_b^0 - T_b\right) \geq C_2\right) < \epsilon, \tag{A.11}$$

for any $C_2 > 0$. Consider the terms of $S_{e,h}(\delta_h, N_b)$ in (A.5). Using $Z_2 = Z_0 \pm Z_\Delta$, the first term can be expanded as

$$\begin{aligned}\delta_h' \left(Z_0' M Z_2\right)\left(Z_2' M Z_2\right)^{-1} Z_2 M e &= \delta_h' \left(\left(Z_2' \pm Z_\Delta\right) M Z_2\right)\left(Z_2' M Z_2\right)^{-1} Z_2 M e \\ &= \delta_h' Z_0' M e \pm \delta_h' Z_\Delta' M e \pm \delta_h' \left(Z_\Delta' M Z_2\right)\left(Z_2' M Z_2\right)^{-1} Z_2 M e. \end{aligned} \tag{A.12}$$

Given Assumption 2.2-(iii), we apply Lemma A.3 to the estimates that do not involve $|N_b - N_b^0|$. Let us focus on the third term,

$$Z_\Delta' M Z_2 / h\left(T_b^0 - T_b\right) = Z_\Delta' Z_2 / h\left(T_b^0 - T_b\right) - \left(Z_\Delta' X_\Delta / \left(h\left(T_b^0 - T_b\right)\right)\left(X'X\right)^{-1} X' Z_2\right). \tag{A.13}$$

Consider $Z_\Delta' Z_\Delta$ (the argument for $Z_\Delta' X_\Delta$ is analogous). By Lemma A.2, $Z_\Delta' Z_\Delta / h\left(T_b^0 - T_b\right)$ uniformly approximates the moving average of $\Sigma_{ZZ,t}$ over the window $(N_b^0 - T^\kappa h,\, N_b^0]$ for large K and small η. Hence, as $h \downarrow 0$, by using the same argument as in (A.10), the first term in (A.13) is

$$Z_\Delta' Z_\Delta / h\left(T_b^0 - T_b\right) = C O_{\mathbb{P}}(1), \tag{A.14}$$

for some $C > 0$, uniformly in T_b. Using Lemma A.3, we deduce that the second term is thus also $O_{\mathbb{P}}(1)$ uniformly. Combining (A.12)-(A.14), we have

$$\begin{aligned} & h^{-1/2}\left(\delta_h\right)'\left(Z_\Delta' M Z_2 / h\left(T_b^0 - T_b\right)\right)\left(Z_2' M Z_2\right)^{-1} Z_2 M e \\ & \leq h^{-1/4} \frac{Z_\Delta' M Z_2}{h\left(T_b^0 - T_b\right)} O_{\mathbb{P}}(1)\, O_{\mathbb{P}}\left(h^{(5/4-\kappa)\wedge 1/2}\right) = O_{\mathbb{P}}\left(h^{1/4}\right), \end{aligned} \tag{A.15}$$

where $\left(Z_2' M Z_2\right)^{-1} = O_{\mathbb{P}}(1)$ and the term $O_{\mathbb{P}}(h^{(5/4-\kappa)\wedge 1/2})$ term follows from equation (S.33) in CP (2020a). So the right-and side of (A.15) is less than $\varepsilon/4$ in probability for large T. The second term of (A.12) can be dealt with as in equation (S.41) in CP (2020a). We deduce that the second term of (A.12) is such that

$$\begin{aligned} K^{-1} h^{-(1-\kappa)-1/2} \delta_h' Z_\Delta' M e &= \frac{h^{-1/2}}{K h^{1-\kappa}} \delta_h' \sum_{k=T_b+1}^{T_b^0} z_{kh} e_{kh} - \frac{h^{-1/2}}{h^{1-\kappa}} \delta_h' \left(\sum_{k=T_b+1}^{T_b^0} z_{kh} x_{kh}'\right)\left(X'X\right)^{-1}\left(X'e\right) \\ &\leq \frac{h^{-1/2}}{K h^{1-\kappa}} \delta_h' \sum_{k=T_b+1}^{T_b^0} z_{kh} e_{kh} - C \frac{1}{K} \frac{h^{-1/4}}{h^{1-\kappa}}\left(\delta^0\right)\left(\sum_{k=T_b+1}^{T_b^0} z_{kh} x_{kh}'\right)\left(X'X\right)^{-1}\left(X'e\right) \\ &\leq \frac{h^{-1/2}}{K h^{1-\kappa}} \delta_h' \sum_{k=T_b+1}^{T_b^0} z_{kh} e_{kh} - h^{-1/4} O_{\mathbb{P}}(1)\, O_{\mathbb{P}}\left(h^{1/2}\right). \end{aligned} \tag{A.16}$$

Thus, using (A.12), (A.5) can be written as

$$\begin{aligned} & \frac{h^{-1/2}}{K h^{1-\kappa}} 2 \delta_h' Z_0' M e \pm \frac{h^{-1/2}}{K h^{1-\kappa}} 2 \delta_h' Z_\Delta' M e \pm \frac{h^{-1/2}}{K h^{1-\kappa}} 2 \delta_h' \left(Z_\Delta' M Z_2\right)\left(Z_2' M Z_2\right)^{-1} Z_2 M e - 2 \frac{h^{-1/2}}{K h^{1-\kappa}} \delta_h' \left(Z_0' M e\right) \\ & \leq \frac{h^{-1/2}}{K h^{1-\kappa}} \left(\delta^0\right)' \sum_{k=T_b+1}^{T_b^0} z_{kh} \widetilde{e}_{kh} \pm O_{\mathbb{P}}\left(h^{1/4}\right), \end{aligned}$$

in view of (A.15) and (A.16). It remains to consider (A.6). We can use the results in CP (2020a). In particular, following the steps from their equation (S.42)-(S.43) to their displayed equation before (S.44), yields that (A.6) is stochastically small uniformly in $T_b \in \mathbf{T}_{K,\eta}$ as $h \downarrow 0$. Combining all the results the relationship in (A.11) holds, concludes the proof. □

A.2 Proofs of the Results of Section 4

The expression for G_h, $\overline{Q}_h$ and $Q_{0,h}$ are given by, respectively, $G_h(N_b) = S_{e,h}(\delta^0, N_b)$, $\overline{Q}_h(N_b) = D_h(N_b) + S_{e,h}(N_b)$ and $Q_{0,h}(N_b) = D_h(N_b)$.

A.2.1 Proof of Lemma 4.1

Proof. We show that for any $\eta > 0$, there exists a $C_2 > 0$ such that

$$Q_h(\delta_h, N_b) - Q_h\left(\delta_h, N_b^0\right) < -C_2, \tag{A.17}$$

for every N_b that satisfies $|N_b - N_b^0| \geq \eta$. Recall the decomposition in (A.8), then (A.17) can be proved by showing that for any $\epsilon_2 > 0$,

$$\mathbb{P}\left(\sup_{|N_b - N_b^0| \geq \eta} S_{e,h}(\delta_h, N_b) \geq \inf_{|N_b - N_b^0| \geq \eta} \left|T_b - T_b^0\right| \overline{D}_h(\delta_h, N_b)\right) < \epsilon_2. \tag{A.18}$$

Suppose that $N_b < N_b^0$. The case with $N_b \geq N_b^0$ can be proved similarly. By definition, note that $|N_b - N_b^0| \geq \eta$ is equivalent to $|T_b - T_b^0| \geq T\eta$. Then,

$$\begin{aligned}
&\mathbb{P}\left(\sup_{|N_b - N_b^0| \geq \eta} h^{-1/2}|S_{e,h}(\delta_h, N_b)| \geq \inf_{|N_b - N_b^0| \geq \eta} h^{-1/2}\left|T_b - T_b^0\right| \overline{D}_h(\delta_h, N_b)\right) \\
&\leq \mathbb{P}\left(D_{\eta,h}^{-1} \sup_{|T_b - T_b^0| \geq T\eta} h^{-1/2}|S_{e,h}(\delta_h, N_b)| \geq \eta\right),
\end{aligned}$$

where $D_{\eta,h} = T\inf_{|T_b - T_b^0| \geq T\eta} h^{-1/2}\overline{D}_h(N_b)$. Lemma A.8 below shows that $D_{\eta,h}$ is positive and bounded away from zero. Thus, it is sufficient to show that $\sup_{|T_b - T_b^0| \geq T\eta} h^{-1/2}|S_{e,h}(\delta_h, N_b)| = o_{\mathbb{P}}(1)$. The latter result was shown to hold by CP (2020a) [cf. (S.31)]. Thus,

$$\mathbb{P}\left(\overline{D}_{\eta,h}^{-1} \sup_{|T_b - T_b^0| \geq T\eta} h^{-1/2}|S_{e,h}(\delta_h, N_b)| \geq \eta\right) < \epsilon_2,$$

which implies (A.18) and concludes the proof. □

Lemma A.8. *For any $B > 0$, let $D_{B,h} = \inf_{|T_b - T_b^0| > TB} Th^{-1/2}\overline{D}_h(\delta_h, N_b)$. There exists an $A > 0$ such that for every $\epsilon > 0$, there exists a $B < \infty$ such that $P(D_{B,h} \geq A) \leq 1 - \epsilon$. That is, $D_{B,h}$ is positive and bounded away from zero with high probability.*

Proof. It follows from Lemma S.A.8 in CP (2020a). □

A.2.2 Proof of Theorem 4.1

We shall first prove a number of preliminary lemmas. We need the following additional notation. Recall that $l \in \boldsymbol{L}$ and thus there exist some real number B sufficiently large and some a sufficiently small such that

$$\inf_{|u|>B} l(u) - \sup_{|u|\leq B^a} l(u) \geq 0. \tag{A.19}$$

Let $\zeta_h(u) = \exp(\gamma_h \widetilde{G}_h(u) - \Lambda^0(u))$, $\Gamma_M = \{u \in \mathbb{R} : M \leq |u| < M+1\} \cap \Gamma_h$ and set

$$J_{1,M} \triangleq \int_{\Gamma_M} \zeta_h(u)\, \pi_h(u)\, du, \qquad J_2 \triangleq \int_{\Gamma_h} \zeta_h(u)\, \pi_h(u)\, du. \tag{A.20}$$

The proof involves showing the weak convergence toward the Gaussian process $\mathscr{V}(u) = \mathscr{W}(u) - \Lambda^0(u)$ where

$$\mathscr{W}(u) = \begin{cases} 2(\delta^0)' \mathscr{W}_1(u), & u \leq 0 \\ 2(\delta^0)' \mathscr{W}_2(u), & u > 0 \end{cases}, \qquad \Lambda^0(u) = \begin{cases} |u|(\delta^0)' \Sigma_{Z,1}\delta^0, & u \leq 0 \\ u(\delta^0)' \Sigma_{Z,2}\delta^0, & u > 0 \end{cases}.$$

Finally, we introduce the following class of functions. The function $f_h : \mathbb{R} \to \mathbb{R}$ is said to belong to the family $\boldsymbol{F}$ if it possesses the following properties: (1) For a fixed h, $f_h(x) \uparrow \infty$ monotocically as $x \uparrow \infty$ and $x \in [0, \infty)$; (2) For any $b < \infty$, $x^b \exp(-f_h(x)) \to 0$ as $h \downarrow 0$, and $x \to \infty$.

For $v \leq 0$ define $Q^*_{0,h}(v) \triangleq -(\delta^0)' (\sum_{k=T^0_b+\lfloor v/h \rfloor}^{T^0_b} z_{kh} z'_{kh})\delta^0$, and $G^*_h(v) \triangleq 2(\delta^0)' \sum_{k=T^0_b+\lfloor v/h \rfloor}^{T^0_b} z_{kh}\widetilde{e}_{kh}$, where $\widetilde{e}_{kh} \sim \mathscr{N}(0, \sigma^2_{e,(k-1)}h)$. Define analogously $Q^*_{0,h}(v)$ and $G^*_h(v)$ for $v > 0$, so that $\widetilde{Q}^*_h(v) = \widetilde{Q}^*_{0,h}(v) + \widetilde{G}^*_h(v)$. The following lemma follows from Proposition 3.2.

Lemma A.9. *Let $v \in \Gamma^*$. If $v \leq 0$, then $Q^*_{0,h}(v) \overset{\text{u.c.p.}}{\Rightarrow} [Z, Z]_1$ where $[Z, Z]_1 \triangleq [Z, Z]_{h\lfloor N^0_b/h \rfloor} - [Z, Z]_{h\lfloor t_v/h \rfloor}$, $t_v \triangleq N^0_b + v$. If $v > 0$, then $Q^*_{0,h}(v) \overset{\text{u.c.p.}}{\Rightarrow} [Z, Z]_2$ where $[Z, Z]_2 \triangleq [Z, Z]_{h\lfloor t_v/h \rfloor} - [Z, Z]_{h\lfloor N^0_b/h \rfloor}$, $t_v \triangleq N^0_b + v$.*

Lemma A.10. *We have $h^{-1/2}G^*_h(v) \Rightarrow \mathscr{W}(v)$ in $\mathbb{D}_b(\mathbf{C})$, where $\mathbf{C} \subset \mathbb{R}$ is a compact set and $\mathscr{W}(v) = ((\delta^0)'\Omega_1\delta^0)^{1/2} W_1(-v)$ if $v < 0$ and $\mathscr{W}(v) = ((\delta^0)'\Omega_2\delta^0)^{1/2} W_2(v)$ if $v > 0$.*

*Proof.*This result also follows from Proposition 3.2 adapted to the case of stationary regimes. Hence, $h^{-1/2}G^*_h(v) \Rightarrow \mathscr{W}(v)$ in $\mathbb{D}_b(\mathbf{C})$. □

Lemma A.11. *On $\{u \leq K\}$, $\gamma_h \widetilde{Q}_h(u) \overset{d}{\equiv} \widetilde{Q}^*_h(v) \Rightarrow \mathscr{V}(v)$, where "$\overset{d}{\equiv}$" signifies equivalence in distribution and $v \in \Gamma^* = (-\vartheta N^0_b, \vartheta(N - N^0_b))$. On the other hand, $\exp(\mathscr{V}(v)) \overset{\mathbb{P}}{\to} 0$ for all $v \notin \Gamma^*$.*

Proof. Assume $u \leq 0$ (i.e., $N_b \leq N^0_b$). We now employ a change of time scale as in CP (2020a). On the old time scale, given $\{|u| \leq K\}$, $N_b(u)$ varies on the time interval $[N^0_b - Kh^{1-\kappa}, N^0_b + Kh^{1-\kappa}]$ for some $K < \infty$. Lemma A.6 shows that the asymptotic behavior of $Q_T(T_b(u)) - Q_T(T^0_b)$ is determined by $-\delta'_h(Z'_\Delta Z_\Delta)\delta_h \pm 2\delta'_h(Z'_\Delta e)$. Next, observe that scaling the criterion function $Q_T(T_b(u)) - Q_T(T^0_b)$ by ψ^{-1}_h has the effect of changing the time scale $s \mapsto t \triangleq \psi^{-1}_h s$. That is, recall the processes Z_s and $\widetilde{e}_s$ as defined in (2.3) and (3.1), respectively, and let $Z_{\psi,s} \triangleq \psi^{-1/2}_h Z_s$, $W_{\psi,e,s} \triangleq \psi^{-1/2}_h W_{e,s}$. Then, for $s \in [N^0_b - Kh^{1-\kappa}, N^0_b + Kh^{1-\kappa}]$,

$$Z_{\psi,s} = \psi^{-1/2}_h \sigma_{Z,s} dW_{Z,s}, \qquad W_{\psi,e,s} = \psi^{-1/2}_h \sigma_{e,s} dW_{e,s}. \tag{A.21}$$

With $s \in [N^0_b - Kh^{1-\kappa}, N^0_b + Kh^{1-\kappa}]$ and $v = \psi^{-1}_h(N^0_b - s)$, using the properties of $W_{\cdot,s}$ and the $\mathscr{F}_s$-measurability of $\sigma_{Z,s}$, $\sigma_{e,s}$, we have

$$Z_{\psi,t} = \sigma_{Z,t} dW_{Z,t}, \qquad W_{\psi,e,t} = \sigma_{e,t} dW_{e,t}, \qquad t \in \left[-\vartheta N^0_b, \vartheta\left(N - N^0_b\right)\right]. \tag{A.22}$$

This results in a change of time scale which we apply to $Z'_\Delta Z_\Delta/\psi_h$ and $Z'_\Delta\widetilde{e}/\psi_h$ so that

$$Z'_\Delta Z_\Delta/\psi_h = \sum_{k=T_b^0+\lfloor v/h\rfloor}^{T_b^0} z_{\psi,kh} z'_{\psi,kh}, \qquad Z'_\Delta\widetilde{e}/\psi_h = \sum_{k=T_b^0+\lfloor v/h\rfloor}^{T_b^0} z_{\psi,kh}\widetilde{e}_{\psi,kh}, \tag{A.23}$$

where $z_{\psi,kh} = z_{kh}/\sqrt{\psi_h}$ and $\widetilde{e}_{\psi,kh} = \widetilde{e}_{kh}/\sqrt{\psi_h}$ with $v \in \Gamma^*$. In view of Condition 1, $\gamma_h \asymp h^{-1/2}\psi_h^{-1}$. By Lemma A.6 and (A.23),

$$\left(Q_h(N_b) - Q_h\left(N_b^0\right)\right)/\psi_h = -\delta_h\left(Z'_\Delta Z_\Delta\right)\delta_h \pm 2\delta'_h\left(Z'_\Delta\widetilde{e}\right) + o_{\mathbb{P}}\left(h^{1/2}\right).$$

or $\gamma_h\overline{Q}_h(N_b) = -\left(\delta^0\right)' Z'_\Delta Z_\Delta\left(\delta^0\right) \pm 2\left(\delta^0\right)'\left(h^{-1/2}Z'_\Delta\widetilde{e}\right) + o_{\mathbb{P}}(1)$ where $e_{kh} \sim$ i.n.d. $\mathscr{N}\left(0,\, \sigma^2_{h,k-1}h\right)$, $\sigma_{h,k} = O(h^{-1/4})\sigma_{e,k}$ and $\widetilde{e}_{kh}$ is the normalized error [i.e., $\widetilde{e}_{kh} \sim$ i.n.d. $\mathscr{N}(0,\, \sigma^2_{e,k-1}h)$] from (3.1). In summary, $\gamma_h\widetilde{Q}_h(u) \overset{d}{\equiv} \widetilde{Q}^*_h(v)$, while by Lemma A.9-A.10, $\widetilde{Q}^*_h(v) \Rightarrow \mathscr{V}(v)$ on Γ^*. For $|u| > K$ and K large enough, we have $\gamma_h\widetilde{Q}_h(v) \downarrow -\infty$ in probability upon using the same arguments as in Lemma A.12-A.13 and thus we use the shorthand notation $\exp(\mathscr{V}(u)) \overset{\mathbb{P}}{\to} 0$ for all $v \notin \Gamma^*$. □

Lemma A.12. *For any $l \in \boldsymbol{L}$ and any $\epsilon > 0$,*

$$\liminf_{h\downarrow 0}\mathbb{P}\left[\int_{K\le|u|\le\eta T^{1-k}} l(s-u)\exp\left(\gamma_h\widetilde{Q}_h(u)\right)\pi\left(N_b^0 + u/r_h\right) < \epsilon\right] \ge 1-\epsilon, \tag{A.24}$$

for sufficiently large K and small $\eta > 0$.

Proof. Since $l \in \boldsymbol{L}$, we have $l(s-u) \le 1 + |s-u|^m$ for $m > 0$. Thus, we shall find a sufficiently large K such that

$$\exp\left(\gamma_h\widetilde{Q}_h(u)\right)\pi_h\left(N_b^0 + u/r_h\right) \overset{\mathbb{P}}{\to} 0, \qquad K \le |u| \le \eta T^{1-k}, \tag{A.25}$$

as $h \downarrow 0$. By Assumption 4.3, $\pi(\cdot)$ satisfies $\pi\left(N_b^0\right) < C_\pi$, for some real number $C_\pi < \infty$. Thus, $\pi(\cdot)$ will play no role in proving (A.25). By Lemma A.7, for large K there exists a $C > 0$ such that

$$\liminf_{h\downarrow 0}\mathbb{P}\left[\sup_{K\le|u|\le\eta T^{1-\kappa}} Q_h(\delta_h,\, N_b) - Q_h\left(\delta_h,\, N_b^0\right) < -C\right] = 1.$$

Consequently, $\gamma_h\widetilde{Q}_h(u)$ diverges to $-\infty$ for values of u satisfying $K \le |u| \le \eta T^{1-\kappa}$. By the property of the exponential function, we have for some finite $C_1,\, C_2 > 0$, $\exp(\gamma_h\widetilde{Q}_h(u)) = C_1\exp(-\kappa_\gamma T^{3/2-\kappa}C_2) \overset{\mathbb{P}}{\to} 0$ for all $K \le |u| \le \eta T^{1-k}$. Combining all the arguments above, the latter result implies (A.24), which concludes the proof. □

Lemma A.13. *For every $l \in \boldsymbol{L}$, and any $\epsilon,\, \eta > 0$, we have*

$$\liminf_{h\downarrow 0}\mathbb{P}\left[\int_{|u|\ge\eta T^{1-\kappa}} l(s-u)\exp\left(\gamma_h\widetilde{Q}_h(u)\right)\pi_h\left(N_b^0 + u/r_h\right) < \epsilon\right] \ge 1-\epsilon. \tag{A.26}$$

Proof. Since $l \in \boldsymbol{L}$, we have $l(s-u) \le 1 + |s-u|^m$ for $m > 0$. We shall show

$$\int_{|u|\ge\eta T^{1-\kappa}}\left(1 + |s-u|^m\right)\left|\exp\left(\gamma_h\widetilde{Q}_h(u)\right)\pi_h\left(N_b^0 + u/r_h\right)\right| du = o_{\mathbb{P}}(1). \tag{A.27}$$

The left-hand side above is no larger than

$$
\begin{aligned}
&C\int_{|u|\geq\eta T^{1-\kappa}}\left|s-T^{1-\kappa}\left(N_b-N_b^0\right)\right|^m\exp\left(\gamma_h\widetilde{Q}_h\left(u\right)\right)\pi_h\left(N_b^0+u/r_h\right)du\\
&\quad\leq C_m\left(T^{1-\kappa}\right)^{m+1}\int_{\left|N_b-N_b^0\right|\geq\eta/2}\left|N_b-N_b^0\right|^m\exp\left(\gamma_h\widetilde{Q}_h\left(u\right)\right)\pi\left(N_b\right)dN_b,
\end{aligned}
$$

where $C_m<\infty$ may depend on m. By Lemma 4.1 there exists a $c>0$ such that

$$
\liminf_{h\downarrow 0}\mathbb{P}\left(\sup_{\left|N_b-N_b^0\right|\geq\eta/2}\exp\left(\gamma_h\widetilde{Q}_h\left(u\right)\right)\leq\exp\left(-c\gamma_h\right)\right)=1.
$$

Then, the left-hand side of (A.27) is bounded by

$$
C_m\left(T^{1-\kappa}\right)^{m+1}e^{-c\gamma_h}\int_{[0,\,N]}N_b^m\pi\left(N_b\right)dN_b=o_{\mathbb{P}}\left(1\right),
$$

since $\exp\left(-c\gamma_h\right)\to 0$ at a rate faster than $\left(T^{1-\kappa}\right)^{m+1}\to\infty$. This concludes the proof. □

Lemma A.14. *Let $u_1,\,u_2\in\mathbb{R}$ being of the same sign and satisfying $0<|u_1|<|u_2|<K<\infty$. For any integer $r>0$ and some constants c_r and C_r which depend on r only, we have*

$$
\mathbb{E}\left[\left(\zeta_h^{1/2r}\left(u_2\right)-\zeta_h^{1/2r}\left(u_1\right)\right)^{2r}\right]\leq c_r\left|\left(\delta^0\right)'\left(\left|u_2-u_1\right|\Sigma_i\right)\delta^0\right|^r\leq C_r\left|u_2-u_1\right|^r,
$$

where $i=1$ if $u_1<0$ and $i=2$ if $u_1>0$.

Proof. The proof is given only for the case $u_2>u_1>0$. We follow Lemma III.5.2 in Ibragimov and Has'minskiĭ (1981). Let $\mathcal{V}\left(u_i\right)=\exp\left(\mathscr{V}\left(u_i\right)\right)$ for $i=1,\,2$ where $\mathscr{V}\left(u_i\right)=\mathscr{W}\left(u_i\right)-\varLambda^0\left(u_i\right)$ and $\mathcal{V}_{u_1}\left(u_2\right)\triangleq\exp\left(\mathscr{V}\left(u_2\right)-\mathscr{V}\left(u_1\right)\right)$. We have

$$
\mathbb{E}\left[\left(\mathcal{V}^{1/2r}\left(u_2\right)-\mathcal{V}^{1/2r}\left(u_1\right)\right)^{2r}\right]=\sum_{j=0}^{2r}\binom{2r}{j}\left(-1\right)^j\mathbb{E}_{u_1}\left[\mathcal{V}_{u_1}^{j/2r}\left(u_2\right)\right].
$$

For any given $u\in\mathbb{R}$, $\mathcal{V}_\cdot\left(u\right)$ is the exponential of a Gaussian random variable, and thus

$$
\mathbb{E}_{u_1}\left[\mathcal{V}_{u_1}^{j/2r}\left(u_2\right)\right]=\exp\left(\frac{1}{2}\left(\frac{j}{2r}\right)^2 4\left(\delta^0\right)'\left(\left|u_2-u_1\right|\Sigma_2\right)\delta^0-\frac{j}{2r}\left|\varLambda^0\left(u_2\right)-\varLambda^0\left(u_1\right)\right|\right). \tag{A.28}
$$

Letting $d\triangleq\exp\left(j\left(2r\right)^{-1}2\left(\delta^0\right)'\left(\left|u_2-u_1\right|\Sigma_2\right)\delta^0-\left|\varLambda^0\left(u_2\right)-\varLambda^0\left(u_1\right)\right|\right)$, we have

$$
\mathbb{E}\left[\left(\mathcal{V}^{1/2r}\left(u_2\right)-\mathcal{V}^{1/2r}\left(u_1\right)\right)^{2m}\right]=\sum_{j=0}^{2r}\binom{2r}{j}\left(-1\right)^j d^{j/2r}.
$$

We need to study three different cases. Let $\varpi\triangleq 2\left(\delta^0\right)'\left(\left|u_2-u_1\right|\Sigma_2\right)\delta^0-\left|\varLambda^0\left(u_2\right)-\varLambda^0\left(u_1\right)\right|$.
(1) $\varpi<0$. Note that

$$
\begin{aligned}
d&=\exp\left(\frac{j}{r}\left(\delta^0\right)'\left(\left|u_2-u_1\right|\Sigma_2\right)\delta^0-2\left|\left(\delta^0\right)'\left(\left|u_2-u_1\right|\Sigma_2\right)\delta^0\right|+\varpi\right)\\
&=\exp\left(-\frac{2r-j}{r}\left(\delta^0\right)'\left(\left|u_2-u_1\right|\Sigma_2\right)\delta^0\right)e^{\varpi},
\end{aligned}
$$

which then yields

$$\mathbb{E}\left[\left(\mathcal{V}^{1/2r}\left(u_{2}\right)-\mathcal{V}^{1/2r}\left(u_{1}\right)\right)^{2r}\right]\leq p_{r}\left(a\right), \tag{A.29}$$

where $p_{r}\left(a\right)\triangleq\sum_{j=0}^{2r}\binom{2r}{j}\left(-1\right)^{j}a^{(2r-j)}$ and $a=\exp\left(-r^{-1}\left(\delta^{0}\right)'\left(\left|u_{2}-u_{1}\right|\Sigma_{2}\right)\delta^{0}\right)$.
(2) $2\left(\delta^{0}\right)'\left(\left|u_{2}-u_{1}\right|\Sigma_{2}\right)\delta^{0}=\left|\Lambda^{0}\left(u_{2}\right)-\Lambda^{0}\left(u_{1}\right)\right|$. This case is the same as the previous one but with $a=\exp\left(-r^{-1}\left(\delta^{0}\right)'\left(\left|u_{2}-u_{1}\right|\Sigma_{2}\right)\delta^{0}\right)$.
(3) $\varpi>0$. Upon simple manipulations, $d=\exp\left(-\left(2r-j\right)r^{-1}\left(\delta^{0}\right)'\left(\left|u_{2}-u_{1}\right|\Sigma_{2}\right)\delta^{0}\right)e^{\varpi}$. Then

$$\mathbb{E}\left[\left(\mathcal{V}^{1/2r}\left(u_{2}\right)-\mathcal{V}^{1/2r}\left(u_{1}\right)\right)^{2r}\right]\leq e^{\varpi/2r}\sum_{j=0}^{2r}\binom{2r}{j}\left(-1\right)^{j}a^{(2r-j)},$$

with $a=\exp\left(-r^{-1}\left(\delta^{0}\right)'\left(\left|u_{2}-u_{1}\right|\Sigma_{2}\right)\delta^{0}\right)$.

We focus on the first case only. It is not difficult to see that the same proof is valid for the other cases. It is enough to show that at the point $a=1$, the polynomial $p_{r}\left(a\right)$ admits a root of multiplicity r. This follows from the equalities $p_{r}\left(1\right)=p_{r}^{(1)}\left(1\right)=\cdots=p_{r}^{(r-1)}\left(1\right)=0$. Then, note that $p_{r}^{(i)}\left(a\right)$ is a linear combination of summations $\mathcal{S}_{k}$ $\left(k=0,\,1,\ldots,\,2i\right)$ given by $\mathcal{S}_{k}=\sum_{j=0}^{2r}\binom{2r}{j}j^{k}$. Thus, all one needs to verify is that that $\mathcal{S}_{k}=0$ for $k=0,\,1,\ldots,\,2r-2$. $\mathcal{S}_{k}$ can be found by applying the k-fold of the operator $a\left(d/da\right)$ to the function $\left(1-a^{2}\right)^{2r}$ and evaluating it at $a=1$. Consequently, $\mathcal{S}_{k}=0$ for $k=0,\,1,\ldots,\,2r-1$. Using this into equation (A.29),

$$\mathbb{E}\left[\left(\mathcal{V}^{1/2r}\left(u_{2}\right)-\mathcal{V}^{1/2r}\left(u_{1}\right)\right)^{2r}\right]\leq\left(1-a\right)^{r}\widetilde{p}_{r}\left(a\right)\leq\left(r^{-1}\left(\delta^{0}\right)'\left(\left|u_{2}-u_{1}\right|\Sigma_{2}\right)\delta^{0}\right)^{r}\widetilde{p}_{r}\left(a\right), \tag{A.30}$$

where $\widetilde{p}_{r}\left(a\right)$ is a polynomial of degree $r^{2}-r$, and the last inequality follows from $1-e^{-c}\leq c$, for $c>0$. Now, write $\overline{\zeta}_{h}^{1/2r}\left(u_{2},\,u_{1}\right)=\zeta_{h}^{1/2r}\left(u_{2}\right)-\zeta_{h}^{1/2r}\left(u_{1}\right)$. By Lemma A.10, the continuous mapping theorem and (A.30), we have $\lim_{h\rightarrow0}\mathbb{E}[(\overline{\zeta}_{h}^{1/2r}\left(u_{2},\,u_{1}\right))^{2r}]\leq\left(1-a\right)^{r}\widetilde{p}_{r}\left(a\right)$. Since $j\leq2r$, we set $c_{r}=\max_{0\leq a\leq1}\widetilde{p}_{r}\left(a\right)/r^{r}<\infty$ and the claim of the lemma follows since $\left\Vert \delta^{0}\right\Vert ,\,\Sigma_{i}<\infty$. □

Lemma A.15. *Let $0<C<\infty$ and $u_{1},\,u_{2}\in\mathbb{R}$ being of the same sign satisfying $0<\left|u_{1}\right|<\left|u_{2}\right|<K<\infty$. Then, for all h sufficiently small, we have*

$$\mathbb{E}\left[\left(\zeta_{h}^{1/4}\left(u_{2}\right)-\zeta_{h}^{1/4}\left(u_{1}\right)\right)^{4}\right]\leq C_{2}\left|u_{2}-u_{1}\right|^{2}. \tag{A.31}$$

Furthermore, for some constant C_{1} as in Lemma A.14 we have

$$\mathbb{P}\left[\zeta_{h}\left(u\right)>\exp\left(-3C_{1}\left|u\right|/2\right)\right]\leq\exp\left(-C_{1}\left|u\right|/4\right). \tag{A.32}$$

Proof. Use Lemma A.14 with $r=2$ to verify (A.31). For (A.32), assume $u>0$. By Markov's inequality and Lemma A.14,

$$\begin{aligned}\mathbb{P}\left[\zeta_{h}\left(u\right)>\exp\left(-3C_{1}\left|u\right|/2\right)\right] & \leq\exp\left(3C_{1}\left|u\right|/4\right)\mathbb{E}\left[\zeta_{h}^{1/2}\left(u\right)\right]\\
 & \leq\exp\left(\frac{3C_{1}\left|u\right|}{4}-\left(\delta^{0}\right)'\left(\left|u\right|\Sigma_{2}\right)\delta^{0}\right)\leq\exp\left(-\frac{C_{1}\left|u\right|}{4}\right).\square
\end{aligned}$$

Lemma A.16. *Under the conditions of Lemma A.15, for any $\varpi>0$ there exists a finite real number c_{ϖ} and a $\overline{h}$ such that for all $h<\overline{h}$, $\mathbb{P}[\sup_{\left|u\right|>M}\zeta_{h}\left(u\right)>M^{-\varpi}]\leq c_{\varpi}M^{-\varpi}$.*

Proof. It can be easily shown to follow from the previous lemma. □

Lemma A.17. *For a small $\epsilon \leq \bar{\epsilon}$, where $\bar{\epsilon}$ may depend on $\pi(\cdot)$, there exists $0 < C < \infty$ such that*

$$\mathbb{P}\left[\int_0^{\epsilon} \zeta_h(u)\,\pi_h(u)\,du < \epsilon\pi^0\right] < C\epsilon^{1/2}. \tag{A.33}$$

Proof. Since $\mathbb{E}(\zeta_h(0)) = 1$ while $\mathbb{E}(\zeta_h(u)) \leq 1$ for sufficiently small $h > 0$,

$$\mathbb{E}\left|\zeta_h(u) - \zeta_h(0)\right| \leq \left(\mathbb{E}\left(\zeta_h^{1/2}(u) + \zeta_h^{1/2}(0)\right)^2 \mathbb{E}\left(\left(\zeta_h^{1/2}(u) - \zeta_h^{1/2}(0)\right)^2\right)\right)^{1/2} \leq C\left|u\right|^{1/2}, \tag{A.34}$$

where we used Lemma A.14 with $r = 1$. By Assumption 4.3, $\left|\pi_h(u) - \pi^0\right| \leq B\left(T^{1-\kappa}\right)^{-1}|u|$, with $B > 0$. Given that $|u| < \bar{\epsilon}$, it holds that $\pi^0/2 < \pi_h(u)$. Thus, for a small $\bar{\epsilon} > 0$, $\int_0^{\epsilon} \zeta_h(u)\,\pi_h(u)\,du > 2^{-1}\pi^0 \int_0^{\epsilon} \zeta_h(u)\,du$. We can then use $\zeta_h(0) = 1$ to yield,

$$\mathbb{P}\left[\int_0^{\epsilon} \zeta_h(u)\,\pi_h(u)\,du < 2^{-1}\epsilon\right] \leq \mathbb{P}\left[\int_0^{\epsilon} \left(\zeta_h(u) - \zeta_h(0)\right)du < -2^{-1}\epsilon\right] \leq \mathbb{P}\left[\int_0^{\epsilon} \left|\zeta_h(u) - \zeta_h(0)\right|du > 2^{-1}\epsilon\right].$$

By Markov's inequality together with inequality (A.34), the right-hand side above is less than or equal to $2\epsilon^{-1}\int_0^{\epsilon} \mathbb{E}\left|\zeta_h(u) - \zeta_h(0)\right|du < 2C\epsilon^{1/2}$. □

Lemma A.18. *For $f_h \in \boldsymbol{F}$, and M sufficiently large, there exist constants c, $C > 0$ such that*

$$\mathbb{P}\left[J_{1,M} > \exp\left(-cf_h(M)\right)\right] \leq C\left(1 + M^C\right)\exp\left(-cf_h(M)\right). \tag{A.35}$$

Proof. Since $u \in \Gamma_h$, and given the differentiability of $\pi(\cdot)$, we consider for simplicity the case of the uniform prior (i.e., $\pi_h(u) = 1$ for all u). We divide the open interval $\{u : M \leq |u| < M + 1\}$ into I disjoint segments and denote the i-th segment by Π_i. Choose a point $u_i \in \Pi_i$ and define $J_{1,M}^{\Pi} \triangleq \sum_{i \in I} \zeta_h(u_i)\,\mathrm{Leb}(\Pi_i) = \sum_{i \in I} \int_{\Pi_i} \zeta_h(u)\,du$. Then,

$$\begin{aligned}
\mathbb{P}\left[J_{1,M}^{\Pi} > 4^{-1}\exp\left(-cf_T(M)\right)\right] &\leq \mathbb{P}\left[\max_{i \in I} \zeta_h^{1/2}(u_i)\left(\mathrm{Leb}(\Gamma_M)\right)^{1/2} > (1/2)\exp\left(-f_h(M)/2\right)\right] \\
&\leq \sum_{i \in I} \mathbb{P}\left[\zeta_h^{1/2}(u_i) > (1/2)\left(\mathrm{Leb}(\Gamma_M)\right)^{-1/2}\exp\left(-f_h(M)/2\right)\right] \\
&\leq 2I\left(\mathrm{Leb}(\Gamma_M)\right)^{1/2}\exp\left(-f_h(M)/12\right),
\end{aligned} \tag{A.36}$$

where the last inequality follows from applying the second part of Lemma A.15 to each summand. For a sufficiently large M, $\exp\left(-f_h(M)/12\right) < 1/2$ and thus,

$$\mathbb{P}\left[J_{1,M} > \exp\left(-f_h(M)/2\right)\right] \leq \mathbb{P}\left[\left|J_{1,M} - J_{1,M}^{\Pi}\right| > 2^{-1}\exp\left(-f_h(M)/2\right)\right] + \mathbb{P}\left[J_{1,M}^{\Pi} > \exp\left(-f_h(M)\right)\right]. \tag{A.37}$$

Let us focus on the first term:

$$\begin{aligned}
\mathbb{E}\left[J_{1,M} - J_{1,M}^{\Pi}\right] &\leq \sum_{i \in I} \int_{\Pi_i} \mathbb{E}\left|\zeta_h^{1/2}(u) - \zeta_h^{1/2}(u_i)\right|du \\
&\leq \sum_{i \in I} \int_{\Pi_i} \left(\mathbb{E}\left|\zeta_h^{1/2}(u) + \zeta_h^{1/2}(u_i)\right|\mathbb{E}\left|\zeta_h^{1/2}(u) - \zeta_h^{1/2}(u_i)\right|\right)^{1/2}du \\
&\leq C(1 + M)^C \sum_{i \in I} \int_{\Pi_i} \left|u_i - u\right|^{1/2}du,
\end{aligned}$$

where the last inequality uses Lemma A.14 and the fact that $u < M + 1$. Note that each summand on the right-hand side above is less than $C\left(MI^{-1}\right)^{3/2}$. Thus, we can find numbers C_1 and C_2 such that

$$\mathbb{E}\left[J_{1,M} - J_{1,M}^{\Pi}\right] \leq C_1\left(1 + M^{C_2}\right) I^{-1/2}. \tag{A.38}$$

Using (A.36) and (A.38) into (A.37), we have

$$\mathbb{P}\left[J_{1,M} > \exp\left(-f_h\left(M\right)/2\right)\right] \leq C_1\left(1 + M^{C_2}\right) I^{-1/2} + 2I\left(\text{Leb}\left(\Gamma_M\right)\right)^{1/2} \exp\left(-f_h\left(M\right)/12\right).$$

The claim of the lemma follows choosing a I that satisfies $1 \leq I^{3/2} \exp\left(-f_h\left(M\right)/12\right) \leq 2$. □

Lemma A.19. *For $f_h \in \boldsymbol{F}$, and M sufficiently large, there exist constants c, $C > 0$ such that*

$$\mathbb{E}\left[J_{1,M}/J_2\right] \leq C\left(1 + M^{C}\right) \exp\left(-cf_h\left(M\right)\right). \tag{A.39}$$

Proof. Note that $J_{1,M}/J_2 \leq 1$ and thus, for an arbitrary $\epsilon > 0$,

$$\mathbb{E}\left[J_{1,M}/J_2\right] \leq \mathbb{P}\left[J_{1,M} > \exp\left(-cf_h\left(M\right)/2\right)\right] + 4\epsilon^{-1}\exp\left(-cf_h\left(M\right)\right) + \mathbb{P}\left[\int_{\Gamma_h} \zeta_h\left(u\right) du < \epsilon/4\right]$$

where we used Lemma A.18 for the first term. In view of the relationship in (A.33), $\mathbb{P}[\int_{\Gamma_h} \zeta_h\left(u\right) du < \epsilon/4] \leq C\epsilon^{1/2}$. To conclude the proof, choose $\epsilon = \exp\left(\left(-2c/3\right) f_h\left(M\right)\right)$. □

Proof of Theorem 4.1. Let $p_{1,h}\left(u\right) \triangleq \widetilde{p}_h\left(u\right)/\overline{p}_h$, where $\widetilde{p}_h\left(u\right) = \exp(\gamma_h(\widetilde{G}_h(u) + \widetilde{Q}_{0,h}(u)))$ and $\overline{p}_h \triangleq \int_{\Gamma_h} p_{1,h}\left(w\right) dw$. By definition, $\widehat{N}_b^{\text{GL}} = N\widehat{\lambda}_b^{\text{GL}}$ minimizes

$$\int_{\Gamma^0} l\left(r_h\left(s - u\right)\right) \exp\left(\gamma_h\left(\widetilde{G}_h\left(u\right) + \widetilde{Q}_{0,h}\left(u\right)\right)\right) \pi\left(u\right) du, \qquad s \in \Gamma^0.$$

Changing variables and using simple manipulations, the expression above is equal to

$$r_h^{-1}\overline{p}_h \int_{\Gamma_h} l\left(r_h\left(s - N_b^0\right) - u\right) p_{1,h}\left(u\right) \pi_h\left(u\right) du, \tag{A.40}$$

from which it follows that $\xi_{l,h} \triangleq \vartheta^{-1}T^{1-\kappa}(\widehat{\lambda}_b^{\text{GL}} - \lambda_0)$ is the minimum of the function

$$\Psi_{l,h}\left(s\right) \triangleq \int_{\Gamma_h} l\left(s - u\right) \frac{\widetilde{p}_h\left(u\right) \pi\left(N_b^0 + u/r_h\right)}{\int_{\Gamma_h} \widetilde{p}_h\left(w\right) \pi\left(N_b^0 + w/r_h\right) dw} du. \tag{A.41}$$

Lemma A.11 shows that, under Condition 1, the normalizing factor γ_h acts as a change of time scale $s \mapsto \psi_h^{-1}s$ since $r_h \asymp \psi_h^{-1}$, where $a_h \asymp b_h$ signifies $b_h/c \leq a_h \leq cb_h$ for some constant c. The change of time scale then implies that the sample criterion $\overline{Q}_h\left(N_b\right)$ is evaluated at $N_b = N_b^0 + \vartheta v$. It also suggests the following change of variable through the substitution $a = \vartheta\psi_h u$,

$$\Psi_{l,h}\left(s\right) = \int_{\Gamma^*} l\left(\left(\vartheta\psi_h\right)^{-1}\left(\vartheta\psi_h s - a\right)\right) \frac{\exp\left(\widetilde{G}_h^*\left(a\right) + \widetilde{Q}_{0,h}^*\left(a\right)\right) \pi\left(N_b^0 + a\right)}{\int_{\Gamma^*} \exp\left(\widetilde{G}_h^*\left(w\right) + \widetilde{Q}_{0,h}^*\left(w\right)\right) \pi\left(N_b^0 + w\right) dw} da, \tag{A.42}$$

where $\Gamma^* = \left(-\vartheta N_b^0, \vartheta\left(N - N_b^0\right)\right)$ and the Quasi-prior is defined on the "fast time scale". This implies that $\Psi_{l,h}^*\left(s\right) \triangleq \Psi_{l,h}\left(\vartheta\psi_h s\right)$ is minimized by $\xi_{l,h} \triangleq Th(\widehat{\lambda}_b^{\text{GL}} - \lambda_0)$. The next step involves showing the finite-dimensional convergence in distribution of the function $\Psi_{l,h}^*\left(s\right)$ to $\Psi_l^*\left(s\right)$ and the tightness of the sequence of functions $\left(\Psi_{l,h}^*\left(s\right)\right)$ on the space $\mathbb{C}_b\left(\Gamma^*\right)$. This gives the weak convergence $\Psi_{l,h}^* \Rightarrow \Psi_l^*$ on

$\mathbb{C}_b(\Gamma^*)$. The final part of the proof requires showing that (a) any element of the set of minimizers of the sample criterion is stochastically bounded, and (b) the length of such set converges to zero as $h \downarrow 0$. Note that we can deduce certain tail properties of $\Psi^*_{l,h}$ from that of $\Psi_{l,h}$ on $\{u \in \mathbb{R}, |u| \leq K\}$ as $K \to \infty$. Thus, we shall sometimes study the behavior of $\Psi_{l,h}$ below. Given the boundedness of $\pi(\cdot)$ [cf. Assumption 4.3], Lemma A.18-A.19 imply that, for any $\nu > 0$, we can find a $\overline{h}$ such that for all $h < \overline{h}$,

$$\mathbb{E}\left[\int_{\Gamma_M} \frac{\exp\left(\gamma_h\left(\widetilde{G}_h(u)+\widetilde{Q}_{0,h}(u)\right)\right)}{\int_{\Gamma_h}\exp\left(\gamma_h\left(\widetilde{G}_h(u)+\widetilde{Q}_{0,h}(u)\right)\right)dw}du\right] \leq \frac{c_\nu}{M^\nu}. \tag{A.43}$$

Furthermore, by Lemma A.12-A.13, for $K < \infty$,

$$\Psi_{l,h}(s) = \int_{\{u\in\mathbb{R},|u|\leq K\}} l(s-u)\frac{\widetilde{p}_h(u)}{\int_{\{u\in\mathbb{R},|u|\leq K\}}\widetilde{p}_h(w)\,dw}du + o_{\mathbb{P}}(1).$$

The last two relationships extend to $\Psi^*_{l,h}(s)$ and in particular (A.43) shows the tail behavior of $\Psi_{l,h}(s)$. On the region $\{|u| \leq K\}$, the change of time scale $u \mapsto \psi_h u = v$ implies that we need to analyze the behavior of $\Psi^*_{l,h}(s)$. Thus, we show the convergence of the marginal distributions of the function $\Psi^*_{l,h}(s)$ to the marginals of $\Psi^*_l(s)$ defined in Definition 4.1. Choose a finite integer n, and arbitrary real numbers a_j $(j = 0,\ldots,n)$. Let $\zeta^*_h(v) = \widetilde{G}^*_h(v) + \widetilde{Q}^*_{0,h}(v)$. The estimate

$$\sum_{j=1}^n a_j \int_{|v|\leq K} l(s_j - v)\,\zeta^*_h(v)\,\pi\left(N^0_b + u\right)dv + a_0\int_{|v|\leq K}\zeta_h(v)\,\pi\left(N^0_b+v\right)dv \tag{A.44}$$

converges in distribution to the random variable

$$\sum_{j=1}^n a_j \int_{|v|\leq K} l(s_j - v)\exp(\mathscr{V}(v))\,\pi\left(N^0_b + v\right)dv + a_0\int_{|v|\leq K}\exp(\mathscr{V}(v))\,\pi\left(N^0_b+v\right)dv,$$

where $\mathscr{V}(v) = \mathscr{W}(v) - \Lambda^0(v)$, since by Lemma A.14-A.15 we can apply Theorem I.A.22 in Ibragimov and Has'minskiĭ (1981). We can then use the Cramer-Wold Theorem [cf. Theorem 29.4 in Billingsley (1995)] to establish the convergence in distribution of the vector

$$\int_{|v|\leq K} l(s_i - v)\,\zeta^*_h(v)\,\pi\left(N^0_b+v\right)dv,\ldots,\int_{|v|\leq K} l(s_n - v)\,\zeta^*_h(v)\,\pi\left(N^0_b+v\right)dv,\ \int_{|v|\leq K}\zeta^*_h(v)\,\pi\left(N^0_b+v\right)dv,$$

to the marginal distributions of the vector

$$\int_{|v|\leq K} l(s_i - v)\exp(\mathscr{V}(v))\,\pi\left(N^0_b+v\right)dv,\ldots,\int_{|v|\leq K} l(s_n - v)\exp(\mathscr{V}(v))\,\pi\left(N^0_b+v\right)dv,$$
$$\int_{|v|\leq K}\exp(\mathscr{V}(v))\,\pi\left(N^0_b+v\right)dv.$$

Note that the above integrals are equal to zero if $v \notin \Gamma^*$, because in the latter result there is an intermediate step involving the change of time scale. That is, $v \notin \Gamma^*$ corresponds to tail behavior as described in (A.43) and the equation right below it. Then, for any K_1, $K_2 < \infty$, the marginal distributions of

$$\frac{\int_{|v|\leq K_1} l(s-v)\exp\left(\widetilde{G}^*_h(v)+\widetilde{Q}^*_{0,h}(v)\right)\pi\left(N^0_b+v\right)dv}{\int_{|w|\leq K_2}\exp\left(\widetilde{G}^*_h(w)+\widetilde{Q}^*_{0,h}(w)\right)\pi\left(N^0_b+w\right)dw},$$

converge to the marginals of $\int_{|v|\leq K_1} l\left(s-v\right)\pi\left(N_b^0+v\right)\exp\left(\mathscr{V}\left(v\right)\right)/\left(\int_{|w|\leq K_2}\exp\left(\mathscr{V}\left(w\right)\right)\pi\left(N_b^0+w\right)dw\right)dv$. By the same reasoning, the marginal distributions of

$$\int_{M\leq|v|<M+1}\frac{\exp\left(\widetilde{G}_h^*\left(v\right)+\widetilde{Q}_{0,h}^*\left(v\right)\right)}{\int_{|w|\leq K_2}\exp\left(\widetilde{G}_h^*\left(w\right)+\widetilde{Q}_{0,h}^*\left(w\right)\right)dw}dv,$$

converge to the distribution of $\int_{M\leq|v|<M+1}(\exp(\mathscr{V}(v)\pi(N_b^0+v))/\int_{|w|\leq K_2}\exp(\mathscr{V}(w))\pi(N_b^0+w)dw)dv$. In view of (A.43) which gives a bound on the tail behavior of the mean of the Quasi-posterior on the original time scale, we have

$$\Psi_{l,h}^*\left(s\right)=\int_{\Gamma^*} l\left(s-v\right)\frac{\exp\left(\mathscr{V}\left(v\right)\right)\pi\left(N_b^0+v\right)dv}{\int_{\Gamma^*}\exp\left(\mathscr{V}\left(w\right)\right)\pi\left(N_b^0+w\right)dw}+o_{\mathbb{P}}\left(1\right). \tag{A.45}$$

This is sufficient for the convergence of the finite-dimensional distributions of $\Psi_{l,h}^*\left(s\right)$ to those of $\Psi_l^*\left(s\right)$. We now turn to the tightness of the sequence of functions $\{\Psi_{l,h}^*(s)\}$. We shall show that the probability distributions in $\mathbb{C}\left([-K,\,K]\right)$ (i.e., the space of continuous functions on $[-K,\,K]$) generated by the contractions of the functions $\Psi_{l,h}^*\left(s\right)$ on $s\in[-K,\,K]$ are tight. For any $l\in\boldsymbol{L}$, we have the inequality $l\left(u\right)\leq 2^r(1+|u|^2)^r$, for some r. Let

$$H_K\left(\varpi\right)\triangleq\int_{\mathbb{R}}\sup_{|s|\leq K,\,|y|\leq\varpi}\left|l\left(s+y-u\right)-l\left(s-u\right)\right|\left(1+|u|^2\right)^{-r-1}du.$$

We show that $\lim_{\varpi\downarrow 0}H_K\left(\varpi\right)=0$. Note that for any $c>0$ we can choose a M such that

$$\int_{|u|>M}\sup_{|s|\leq K,\,|y|\leq\varpi}\left|l\left(s+y-u\right)-l\left(s-u\right)\right|\left(1+|u|^2\right)^{-r-1}du<c.$$

We apply Lusin's theorem [cf. Section 3.3 of Royden and Fitzpatrick (2010)]. Let $\overline{L}$ denote the upper bound of $l\left(\cdot\right)$ on the set $\{u\in\mathbb{R}:\,|u|\leq K+2M\}$. By the measurability of $l\left(\cdot\right)$, we can find a continuous function $J\left(u\right)$ in the interval $\{u:\,|v|\leq K+2M\}$ which equals $l\left(u\right)$ except on a set whose measure does not exceed $c(2\overline{L})^{-1}$. Denote the modulus of continuity of $J\left(\cdot\right)$ by $w_J\left(\varpi\right)$. Without loss of generality assume $|J\left(u\right)|\leq\overline{L}$ on $\{u:\,|u|\leq K+2M\}$. Then,

$$\begin{aligned}&\int_{|u|>M}\sup_{|s|\leq K,\,|y|\leq\varpi}\left|l\left(s+y-u\right)-l\left(s-u\right)\right|\left(1+|u|^2\right)^{-r-1}du\\&\qquad\leq w_J\left(\varpi\right)\int_{\mathbb{R}}\left(1+|u|^2\right)^{-r-1}du+2\overline{L}\text{Leb}\left\{u\in\mathbb{R}:\,|u|\leq K+2M,\,l\neq J\right\},\end{aligned}$$

and $\overline{L}\leq Cw_J\left(\varpi\right)+c$ for some $C<\infty$. Accordingly, $H_K\left(\varpi\right)\leq Cw_J\left(\varpi\right)+2c$, with the property that c can be chosen arbitrary small and $w_J\left(\varpi\right)\to 0$ as $\varpi\downarrow 0$ (holding for each fixed c) by definition. Definition 4.1 implies that we can find a number $C<\infty$ such that

$$\begin{aligned}&\mathbb{E}\left[\sup_{|s|\leq K,\,|y|\leq\varpi}\left|\Psi_{l,h}^*\left(s+y\right)-\Psi_{l,h}^*\left(s\right)\right|\right]\\&\quad\leq\int_{\mathbb{R}}\sup_{|s|\leq K,\,|y|\leq\varpi}\left|l\left(s+y-u\right)-l\left(s-u\right)\right|\mathbb{E}\left(\frac{\zeta_h\left(u\right)\pi_h\left(u\right)}{\int_{\Gamma_h}\zeta_h\left(w\right)\pi_h\left(w\right)dw}\right)du\leq CH_K\left(\varpi\right).\end{aligned}$$

The tightness of $\Psi_{l,h}^*\left(s\right)$ on $s\in\Gamma^*$ is established by using Markov's inequality together with the above bound. It remains to study the oscillations of the minimum points of the sample function $\Psi_{l,h}^*$. Consider an open bounded interval $\mathbf{A}$ satisfying $\mathbb{P}\left[\xi_l^0 \in b(\mathbf{A})\right] = 0$, where $b(\mathbf{A})$ denotes the boundary of the set $\mathbf{A}$. Define the functionals $H_{\mathbf{A}}(\Psi) = \inf_{s\in\mathbf{A}} \Psi_l^*(s)$ and $H_{\mathbf{A}^c}(\Psi) = \inf_{s\in\mathbf{A}^c} \Psi_l^*(s)$, where $\mathbf{A}^c$ is the set complementary to $\mathbf{A}$. Let $\mathbf{M}_h$ denote the set of absolute minimum points of the function $\Psi_{l,h}^*(s)$. By the definition of $\widehat{\lambda}_b^{\mathrm{GL}}$ we have that $\limsup_{h\downarrow 0} \mathbb{E}[l(Th(\widehat{\lambda}_b^{\mathrm{GL}} - \lambda_0))] < \infty$. This implies that any minimum point $\xi \in \mathbf{M}_h$ of the sample criterion function is uniformly stochastically bounded, i.e.,

$$\lim_{K\to\infty} \mathbb{P}\left[\mathbf{M}_h \not\subseteq \{s:\, |s| \leq K\}\right] = 0. \tag{A.46}$$

Next, note that

$$\begin{aligned}\mathbb{P}\left[\mathbf{M}_h \subset \mathbf{A}\right] &= \mathbb{P}\left[H_{\mathbf{A}}(\Psi) < H_{\mathbf{A}^c}(\Psi),\, \mathbf{M}_h \subset \{s:\, |s| \leq K\}\right] \\ &\geq \mathbb{P}\left[H_{\mathbf{A}}(\Psi) < H_{\mathbf{A}^c}(\Psi)\right] - \mathbb{P}\left[\mathbf{M}_h \not\subseteq \{s:\, |s| \leq K\}\right].\end{aligned}$$

Furthermore, the relationships

$$\liminf_{h\downarrow 0} \mathbb{P}\left[\mathbf{M}_h \subset \mathbf{A}\right] \geq \mathbb{P}\left[H_{\mathbf{A}}(\Psi) < H_{\mathbf{A}^c}(\Psi)\right] - \sup_h \mathbb{P}\left[\mathbf{M}_h \not\subseteq \{s:\, |s| \leq K\}\right],$$

and $\limsup_{h\downarrow 0} \mathbb{P}\left[\mathbf{M}_h \subset \mathbf{A}\right] \leq \mathbb{P}\left[H_{\mathbf{A}}(\Psi) < H_{\mathbf{A}^c}(\Psi)\right]$ are valid. As for the minimum point of the population criterion $\Psi_l^*(\cdot)$, we have $\mathbb{P}\left[\xi_l^0 \in \mathbf{A}\right] \leq \mathbb{P}\left[H_{\mathbf{A}}(\Psi) < H_{\mathbf{A}^c}(\Psi)\right]$, and $\mathbb{P}\left[\xi_l^0 \in \mathbf{A}\right] + \mathbb{P}\left[|\xi_l^0| > K\right] \geq \mathbb{P}\left[H_{\mathbf{A}}(\Psi) \leq H_{\mathbf{A}^c}(\Psi)\right]$. The uniqueness assumption on the population criterion (cf. Assumption 4.2) and (A.46) leads to $\lim_{K\to\infty}\{\sup_{h\leq\overline{h}} \mathbb{P}\left[\mathbf{M}_h \not\subseteq \{s:\, |s| \leq K\}\right] + \mathbb{P}\left[|\xi_l^0| > K\right]\} = 0$, for a small $\overline{h} > 0$. Hence,

$$\lim_{h\downarrow 0} \mathbb{P}\left[\mathbf{M}_h \subset \mathbf{A}\right] = \mathbb{P}\left[\xi_l^0 \in \mathbf{A}\right]. \tag{A.47}$$

In the last step of the proof, we show that the length of the set $\mathbf{M}_h$ converges to zero in probability as $h \downarrow 0$. Let $\mathbf{A}_d$ denote an interval in $\mathbb{R}$ centered at the origin and of length $d < \infty$. In view of (A.47), $\lim_{d\to\infty} \sup_{h\downarrow 0} \mathbb{P}\left[\mathbf{M}_h \not\subseteq \mathbf{A}_d\right] = 0$. Fix any $\epsilon > 0$ and divide $\mathbf{A}_d$ into admissible subintervals whose lengths do not exceed $\epsilon/2$. We have,

$$\mathbb{P}\left[\sup_{s_i,s_j\in\mathbf{M}_h} |s_i - s_j| > \epsilon\right] \leq \mathbb{P}\left[\mathbf{M}_h \not\subseteq \mathbf{A}_d\right] + (1 + 2d/\epsilon) \sup \mathbb{P}\left[H_{\mathbf{A}}\left(\Psi_{l,h}^*\right) = H_{\mathbf{A}^c}\left(\Psi_{l,h}^*\right)\right],$$

where $1 + 2d/\epsilon$ is an upper bound on the number of subintervals and the supremum is taken over all possible open bounded subintervals $\mathbf{A} \subset \mathbf{A}_d$. Given $\Psi_{l,h}^* \Rightarrow \Psi_l^*$, we have $\mathbb{P}[H_{\mathbf{A}}(\Psi_{l,h}^*) = H_{\mathbf{A}^c}(\Psi_{l,h}^*)] \to \mathbb{P}[H_{\mathbf{A}}(\Psi) = H_{\mathbf{A}^c}(\Psi)]$ as $h \downarrow 0$. Since $\mathbb{P}\left[H_{\mathbf{A}}(\Psi) = H_{\mathbf{A}^c}(\Psi)\right] = 0$ and d can be chosen large, we have $\mathbb{P}[\sup_{s_i,s_j\in\mathbf{M}_h} |s_i - s_j| > \epsilon] = o(1)$. Given that $\epsilon > 0$ can be chosen arbitrary small we have that the distribution of $\xi_{l,h} = Th(\widehat{\lambda}_b^{\mathrm{GL}} - \lambda_0)$ converges to the distribution of ξ_l^0. □

A.3 Proofs of Section 6

A.3.1 Proof of Theorem 6.1

The next tree lemmas correspond to Lemma A.29-A.31 from Casini and Perron (2020c), respectively.

Lemma A.20. *For* $\varpi > 3/4$, *we have* $\lim_{h\downarrow 0} \limsup_{|s|\to\infty} \left|\widehat{\mathscr{W}}_h(s)\right| / |s|^{\varpi} = 0$ $\mathbb{P}$*-a.s.*

Lemma A.21. $\{\widehat{\mathscr{W}}_h\}$ *converges weakly toward* $\mathscr{W}$ *on compact subsets of* $\mathbb{D}_b$.

Lemma A.22. *Fix $0 < a < \infty$. For $l \in \boldsymbol{L}$ and any positive sequence $\{a_h\}$ satisfying $a_h \overset{\mathbb{P}}{\to} a$,*

$$\int_{\mathbb{R}} |l(r-s)| \exp\left(\widehat{\mathscr{W}}_h(s)\right) \exp(-a_h |s|)\, ds \overset{d}{\to} \int_{\mathbb{R}} |l(r-s)| \exp((\mathscr{W}(s))) \exp(-a|s|)\, ds.$$

Proof of Theorem 6.1. Let $\mathbf{C} \subset \mathbb{R}$ be compact. Suppose γ_h satisfies Condition 1. Then,

$$\widehat{\Psi}^*_{l,h}(s) = \int_{\widehat{\Gamma}^*} l(s-v) \frac{\exp\left(\widehat{\mathscr{V}}_h(v)\right) \pi\left(\widehat{N}_b^{\mathrm{GL}} + v\right)}{\int_{\widehat{\Gamma}^*} \exp\left(\widehat{\mathscr{V}}_h(w)\right) \pi\left(\widehat{N}_b^{\mathrm{GL}} + w\right) dw} dv + o_{\mathbb{P}}(1).$$

Using Lemma A.21, $\widehat{\mathscr{V}}_h \Rightarrow \mathscr{V}$ in $\mathbb{D}_b(\mathbf{C})$. From Assumption 6.1-(i), $\widehat{\Gamma}^*$ can be replaced by Γ^* for small enough h if we show that the integral as a function of $\widehat{N}_b^{\mathrm{GL}}$ is continuous. Note that the integrand is Riemann integrable and thus the integral considered as a map is continuous. Since $\mathscr{V}$ is $\mathbb{P}$-a.s. continuous, then the desired result follows by the continuity of the composition of continuous functions. Using the same steps as in the proof of Theorem 4.1 together with Lemma A.22 yields the result. □

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