Supplement to Continuous Record Laplace-based Inference about the Break Date in Structural Change Models

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Abstract

This supplemental material includes the Mathematical Appendix to Casini and Perron (2020b), which includes all proofs of the results in the paper.

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A Mathematical Appendix

The supplement is structured as follows. In Section A.1 we present some preliminary lemmas. The proof of the results of Section 4 can be found in Section A.2 while those about inference are in Section A.3.

A.1 Additional Notation and Preliminary Results

We denote the \((i, j)\)-th element of the outer product matrix \(A' A\) by \((A' A)_{i,j}\) and the \(i \times j\) upper-left (resp., lower-right) sub-block of \(A' A\) by \([A']_{(i \times j)}\) (resp., \([A]_{(i \times j)}\)). For a random variable \(\xi\) and a number \(r \geq 1\), we write \(\|\xi\|_r = (E|\xi|^r)^{1/r}\). \(C\) is used as a generic constant that may vary from line to line; we may sometime write \(C_r\) to emphasize the dependence of \(C\) on some number \(r\). For two scalars \(a\) and \(b\) the symbol \(a \wedge b\) means the infimum of \(\{a, b\}\). The symbol \("\wedge\leq\geq\)\) signifies uniform locally in time convergence under the Skorokhod topology, which implies convergence in probability.

It is typical in the high frequency statistics literature to use a localization argument [cf. Section I.1.d in Jacod and Shiryaev (2003)] which allows us to replace Assumption 2.1-2.2 by the following stronger assumption which basically turns all the local conditions into global ones.

**Assumption A.1.** Under Assumption 2.1-2.6, \([Y_t, D_t, Z_t]_{t \geq 0}\) takes value in some compact set, \([\sigma_{t}, \sigma_{\epsilon,t}]_{t \geq 0}\) is bounded càdlàg and \([\mu_{t}]\) are bounded càdlàg or càglàd.

Next, we collect a few lemmas. They follow from Lemma S.A.1-S.A.4 in CP (2020a).

**Lemma A.1.** The following inequalities hold \(\mathbb{P}\)-a.s.:

\[
\begin{align*}
(Z_0' M Z_0) - (Z_0' M Z_0) (Z_2' M Z_2)^{-1} & \geq H' (X_\Delta X_\Delta) (X_2' X_2)^{-1} (X_0' X_0) H, \quad T_0 < T_0^0 \quad (A.1) \\
(Z_0' M Z_0) - (Z_0' M Z_0) (Z_2' M Z_2)^{-1} & \geq H' (X_\Delta X_\Delta) (X' X - X_2' X_2)^{-1} (X_0' X_0) H, \quad T_0 \geq T_0^0 \quad (A.2)
\end{align*}
\]

**Lemma A.2.** Let \(X_t (\tilde{X}_t)\) be a \(q\)-dimensional (resp. \(p\)-dimensional) Itô continuous semimartingale defined on \([0, N]\). Let \(\Sigma_t\) denote the date \(t\) instantaneous covariation between \(X_t\) and \(\tilde{X}_t\). Choose a fixed number \(\epsilon > 0\) and \(\varpi\) satisfying \(1/2 - \epsilon > \varpi \geq \epsilon > 0\). Further, let \(B_T \triangleq |N/h - T^{[\varpi]}|\). Define the moving average of \(\Sigma_t\) as \(\Sigma_{kh} \triangleq (T^{[\varpi]} h)^{-1} \sum_{kh} \Sigma_{kh} ds\). Let \(\tilde{\Sigma}_{kh} \triangleq (T^{[\varpi]} h)^{-1} \sum_{i=1}^{kh} \Delta h X_{k+i} \tilde{X}_{k+i}^\prime\). Then, \(\sup_{1 \leq k \leq B_T} \|\Sigma_{kh} - \tilde{\Sigma}_{kh}\| = \mathcal{O}_p(1)\).

**Lemma A.3.** Under Assumption A.1 we have as \(h \downarrow 0, T \to \infty\) with \(N\) fixed and for any \(1 \leq i, j \leq p\),

\[
\begin{align*}
&\text{(i)} \quad \left|Z_{0'}(e)_{i,1}\right| = \sum_{k=T_0^0}^{T_0} z_{kh} e_{kh} \to 0 \quad \text{a.s.} \\
&\text{(ii)} \quad \left|Z_0 Z_{0'}(e)_{i,1}\right| = \sum_{k=T_0^0}^{T_0} z_{kh} e_{kh} \to 0 \quad \text{a.s.} \\
&\text{(iii)} \quad \left|Z_0 Z_0 (Z_0')_{i,1}\right| = \sum_{k=T_0^0}^{T_0} z_{kh} e_{kh} \to 0 \quad \text{a.s.} \\
&\text{(iv)} \quad \left|Z_0 Z_0 (Z_0')_{i,1}\right| = \sum_{k=T_0^0}^{T_0} z_{kh} e_{kh} \to 0 \quad \text{a.s.}
\end{align*}
\]

For the following estimates involving \(X\), we have, for any \(1 \leq r \leq p\) and \(1 \leq l \leq q + p\),

\[
\begin{align*}
&\text{(v)} \quad \left|X (X)'_{1,1}\right| \to 0 \\
&\text{(vi)} \quad \left|X (X)'_{1,1}\right| \to 0 \\
&\text{(vii)} \quad \left|X (X)'_{1,1}\right| \to 0 \\
&\text{(viii)} \quad \left|X (X)'_{1,1}\right| \to 0
\end{align*}
\]

Denote

\[
\begin{align*}
X_\Delta & \triangleq X_2 - X_0 = \left(0, \ldots, 0, x_{(T_0^0+1)h}, \ldots, x_{(T_0^0+1)h}, 0, \ldots, 0\right)' \\
X_\Delta & \triangleq -(X_2 - X_0) = \left(0, \ldots, 0, x_{(T_0^0+1)h}, \ldots, x_{(T_0^0+1)h}, 0, \ldots, 0\right)'
\end{align*}
\]

for \(N_0 < N_0^0\) and \(N_0 > N_0^0\).
whereas $X_\Delta \triangleq 0$ for $N_b = N_b^0$. Observe that $X_2 = X_0 + X_\Delta \text{sign} (N_b^0 - N_b)$. When the sign is immaterial, we simply write $X_2 = X_0 + X_\Delta$.

**Lemma A.4.** Under Assumption A.1, we have as $h \downarrow 0$, $T \to \infty$ with $N$ fixed, $|N_b^0 - N_b| > \gamma > 0$ and for any $1 \leq i, j \leq p$,

(i) with $(Z'_\Delta Z_\Delta)_{i,j} = \sum_{k=T_b^{0-1}+1}^{T_b} z^{(i)}_{kh} z^{(j)}_{kh}$ then

\[
\begin{cases}
(Z'_\Delta Z_\Delta)_{i,j} - \int_{T_b^{(h+1)}}^{T_b} \Sigma_{Z, Z,s} ds \to 0, & \text{if } T_b < T_b^0, \\
(Z'_\Delta Z_\Delta)_{i,j} - \int_{T_b^{(h+1)}}^{T_b} \Sigma_{Z, Z,s} ds \to 0, & \text{if } T_b > T_b^0,
\end{cases}
\]

and for $1 \leq r \leq p+q$

(ii) with $(Z'_\Delta X_\Delta)_{i,r} = 0$

\[
\begin{cases}
(Z'_\Delta X_\Delta)_{i,r} - \int_{0}^{h} \Sigma_{Z, Z,s} ds \to 0, & \text{if } T_b < T_b^0, \\
(Z'_\Delta X_\Delta)_{i,r} - \int_{0}^{h} \Sigma_{Z, Z,s} ds \to 0, & \text{if } T_b > T_b^0.
\end{cases}
\]

Moreover, each quantity on the right hand side above are bounded in probability.

We will also use the following Central Limit Theorem [see Theorem 5.4.2. in Jacod and Protter (2012)]. We choose a progressively measurable “square-root” process $Y_Z$ of the $\mathbb{M}_2^+$-valued process $\tilde{\Sigma}_{Z,s}$ whose elements are given by $\tilde{\Sigma}_{Z,s} = \Sigma_{Z,s}^2 Z_{s}$. Note that by the symmetry of $\Sigma_{Z,s}$, the matrix with entries $(\tilde{\Sigma}_{Z,s} + \tilde{\Sigma}_{Z,s})/\sqrt{2}$ is a square-root of the matrix with entries $\tilde{\Sigma}_{Z,s} + \tilde{\Sigma}_{Z,s}$. Let $W_s^a$ a $q$-dimensional standard Wiener process defined on an extension of $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$. Then the process $\mathcal{U}_t^{(\gamma)} = 2^{-1/2} \sum_{k,t=1}^{\gamma} \int_0^h (\tilde{\Sigma}_{Z,s}^{(ij)} + \tilde{\Sigma}_{Z,s}^{(ij)})dW_s^{(kl)}$ is, conditionally on $\mathcal{F}$, a continuous Gaussian process with independent increments and (conditional) covariance $E(\mathcal{U}_t^{(\gamma)}(v)\mathcal{U}_t^{(\gamma)}(v)|\mathcal{F}) = \int_{N_b^0+\gamma}^{N_b^0+\gamma} (\Sigma_{Z,s}^{(ij)} + \Sigma_{Z,s}^{(ij)}) ds$, with $v < 0$. The Central Limit Theorem we shall use is the following [cf. Lemma S.A.5 in CP (2020a)].

**Lemma A.5.** Let $Z$ be a continuous Itô semimartingale satisfying Assumption A.1. Then, $h^{-1/2}(Z'_2 Z_2 - ([Z, Z]_{(T_b+1)h})) \overset{\mathbb{P}}{\to} \mathcal{U}$.

Recall $Q_h(\tilde{\delta}(N_b), N_b) = \tilde{\delta}(N_b)'(Z'_Z M_Z Z)_2 \tilde{\delta}(N_b)$. We decompose $Q_h(\tilde{\delta}(N_b), N_b) - Q_h(\tilde{\delta}(N_b^0), N_b^0)$ into a “deterministic” and a “stochastic” component. By definition,

\[
\tilde{\delta}(N_b) = (Z'_Z M_Z Z)^{-1} (Z'_Z M_Y) = (Z'_Z M_Z Z)^{-1} (Z'_Z M_Z Z_0) \delta^0 + (Z'_Z M_Z Z)^{-1} Z_2 M e,
\]

and $\tilde{\delta}(N_b^0) = (Z'_0 M_Z Z)^{-1} (Z'_0 M_Y) = \delta^0 + (Z'_0 M_Z Z)^{-1} (Z'_0 M e)$. Therefore,

\[
Q_h(\tilde{\delta}(N_b), N_b) - Q_h(\tilde{\delta}(N_b^0), N_b^0) = \tilde{\delta}(N_b) (Z'_Z M_Z Z_2) \tilde{\delta}(N_b) - \tilde{\delta}(N_b^0) (Z'_0 M Z_0) \tilde{\delta}(N_b^0)
\]

\[
\Delta = D_h \left(\delta^0, N_b\right) + S_{e,h} \left(\delta^0, N_b\right),
\]

where

\[
D_h \left(\delta^0, N_b\right) \triangleq \left(\delta^0\right)' \left(\begin{array}{c} (Z'_0 M Z_2) (Z'_Z M_Z Z_2)^{-1} (Z'_Z M Z_0) - (Z'_0 M Z_0) \end{array}\right)^{\delta^0}
\]

is the deterministic part and

\[
S_{e,h} \left(\delta^0, N_b\right) \triangleq 2 \left(\delta^0\right)' (Z'_0 M Z_2) (Z'_Z M_Z Z_2)^{-1} Z_2 M e - 2 \left(\delta^0\right)' (Z'_0 M e)
\]

\[
+ e' M Z_2 (Z'_Z M_Z Z)^{-1} Z_2 M e - e' M Z_0 (Z'_0 M Z_0)^{-1} Z_0 M e
\]

is the stochastic part. Next, let $Z_\Delta = X_\Delta H$, and define

\[
\overline{D}_h \left(\delta^0, N_b\right) \triangleq -D_h \left(\delta^0, N_b\right) / |T_b - T_b^0|.
\]
We arbitrarily define $\mathcal{D}_h (\delta^0, N_b) = (\delta^0)' \delta^0$ when $N_b = N_b^0$ since both the numerator and denominator of $\mathcal{D}_h (\delta^0, N_b)$ are zero. Observe that $\mathcal{D}_h (\delta^0, N_b)$ is non-negative because the matrix inside the braces in $D_h (\delta^0, N_b)$ is negative semidefinite. Equation (A.3) can be rewritten as

$$Q_h (\delta (N_b), N_b) - Q_h \left( \delta \left( N_b^0 \right), N_b^0 \right) = - \left| T_b - T_b^0 \right| \mathcal{D}_h \left( N_b, \delta^0 \right) + S_{e,h} \left( N_b, \delta^0 \right) \quad \text{for all } T_b. \quad \text{(A.8)}$$

**Lemma A.6.** Under Assumption 2.2-2.6 and 3.1, uniformly in $N_b$,

$$Q_h (\delta (N_b), N_b) - Q_h \left( \delta \left( N_b^0 \right), N_b^0 \right) = - \delta_h (Z_{\Delta} Z_{\Delta}) \delta_h + 2 \delta_h (Z_{\Delta} \tilde{c}) \text{sgn} \left( T_b^0 - T_b \right) + o_P \left( h^{3/2 - \kappa} \right). \quad \text{(A.9)}$$

**Proof.** It follows from Lemma 4.1 in CP (2020a). \(\square\)

**Lemma A.7.** For any $\epsilon > 0$ there exists a $C > 0$ such that

$$\liminf_{h \downarrow 0} \mathbb{P} \left[ \sup_{K \leq |u| \leq \eta T^{1-\kappa}} Q_h \left( N_b^0 + u/r_h \right) - Q_h \left( N_b^0 \right) < -C \right] \geq 1 - \epsilon,$$

for some large $K$ and small $\eta > 0$.

**Proof.** Note that on $\{ K \leq |u| \leq \eta T^{1-\kappa} \}$ we have $KT^a \leq |T_b - T_b^0| < \eta T^b$. Suppose $N_b < N_b^0$. Let $T_{K,\eta} = \{ T_b : \eta N \geq |N_b - N_b^0| > K (T^{1-\kappa})^{-1} \}$. It is enough to show that

$$\mathbb{P} \left( \sup_{T_b \in T_{K,\eta}} Q_h \left( N_b^0 + u/r_h \right) - Q_h \left( N_b^0 \right) \geq 0 \right) < \epsilon,$$

or using (A.8), $\mathbb{P} \left( \sup_{T_b \in T_{K,\eta}} h^{-3/2} S_{e,h} (\delta_h, N_b) / |T_b - T_b^0| \geq \inf_{T_b \in T_{K,\eta}} h^{-3/2} \mathcal{D}_h (\delta_h, N_b) \right) < \epsilon$. The difficulty is to control the estimates that depend on the difference $|N_b - N_b^0|$. By Lemma A.1,

$$\inf_{T_b \in T_{K,\eta}} \mathcal{D}_h (\delta_h, N_b) \geq \inf_{T_b \in T_{K,\eta}} \delta_h H \left( X_{\Delta} X_{\Delta} / |T_b - T_b^0| \right) (X'_2 X_2)^{-1} (X'_0 X_0) H \delta_h,$$

and, since $T \eta > |T_b - T_b^0| \geq KT^a / N$, we need to study the behavior of $X_{\Delta} X_{\Delta} = \sum_{k=T_b}^{T_b+1} x_{kh} x_{kh}'$. We shall apply asymptotic results for the local approximation of the covariation between processes. By Theorem 9.3.2 part (i) in Jacod and Protter (2012),

$$\left( h \left( T_b^0 - T_b \right) \right)^{-1} \sum_{k=T_b+1}^{T_b^0} x_{kh} x_{kh}' \xrightarrow{P} \Sigma_{X,X,N_b^0}, \quad \text{(A.10)}$$

since $|N_b - N_b^0|$ shrinks at a rate no faster than $h^{1-\kappa}$ and $(1/h^{1-\kappa}) \to \infty$. Since $X'_0 X_0$ and $X'_2 X_2$ both involve at least a fixed positive fraction of the data, we can apply a simple law of large numbers for approximate covariation processes [cf. Lemma A.3] to show that $X'_0 X_0$ and $X'_2 X_2$ are each $O_p (1)$. By Lemma A.2, the approximation in (A.10) is uniform, and thus

$$h^{-1/2} \inf_{T_b \in T_{K,\eta}} \delta_h H \left( X_{\Delta} X_{\Delta} / h \right) \left( X'_2 X_2 \right)^{-1} (X'_0 X_0) H \delta_h$$

$$= \inf_{T_b \in T_{K,\eta}} \left( \delta^0 \right)' H \left( X_{\Delta} X_{\Delta} / h \right) \left( X'_2 X_2 \right)^{-1} (X'_0 X_0) H \delta^0$$

is bounded away from zero, in view of Assumption 2.2-(iii). It remains to show that

$$\mathbb{P} \left( \sup_{T_b \in T_{K,\eta}} h^{-3/2} S_{e,h} (\delta_h, N_b) / \left( T_b^0 - T_b \right) \geq C_2 \right) < \epsilon, \quad \text{(A.11)}$$
for any $C_2 > 0$. Consider the terms of $S_{e,h} (\delta_h, N_b)$ in (A.5). Using $Z_2 = Z_0 \pm Z_\Delta$, the first term can be expanded as

$$
\delta_h' (Z'_0 M Z_2) (Z'_2 M Z_2)^{-1} Z_2 M e = \delta_h' ((Z'_2 \pm Z_\Delta) M Z_2) (Z'_2 M Z_2)^{-1} Z_2 M e \\
= \delta_h' Z'_0 M e \pm \delta_h' Z'_\Delta M e \pm \delta_h' (Z'_\Delta M Z_2) (Z'_2 M Z_2)^{-1} Z_2 M e.
$$

(A.12)

Given Assumption 2.2-(iii), we apply Lemma A.3 to the estimates that do not involve $|N_b - N_0^b|$. Let us focus on the third term,

$$
Z'_\Delta M Z_2/h \left( T_0^0 - T_b \right) = Z'_\Delta Z_2/h \left( T_0^0 - T_b \right) - \left( Z'_\Delta X_\Delta / \left( h \left( T_0^0 - T_b \right) \right) \right) (X'X)^{-1} (X'Z_2).
$$

(A.13)

Consider $Z'_\Delta Z_\Delta$ (the argument for $Z'_\Delta X_\Delta$ is analogous). By Lemma A.2, $Z'_\Delta Z_\Delta/h \left( T_0^0 - T_b \right)$ uniformly approximates the moving average of $\Sigma Z Z_t$ over the window $(N_0^b - T^\kappa h, N_0^b)$ for large $K$ and small $\eta$. Hence, as $h \downarrow 0$, by using the same argument as in (A.10), the first term in (A.13) is

$$
Z'_\Delta Z_\Delta/h \left( T_0^0 - T_b \right) = \mathcal{O}_P \left( 1 \right),
$$

(A.14)

for some $C > 0$, uniformly in $T_b$. Using Lemma A.3, we deduce that the second term is thus also $\mathcal{O}_P \left( 1 \right)$ uniformly. Combining (A.12)-(A.14), we have

$$
h^{-1/2} \left( \frac{\delta_h' Z'_0 M Z_2}{h \left( T_0^0 - T_b \right)} \right) (Z'_2 M Z_2)^{-1} Z_2 M e \\
\leq h^{-1/4} \frac{Z'_2 M Z_2}{h \left( T_0^0 - T_b \right)} \mathcal{O}_P \left( 1 \right) \mathcal{O}_P \left( h^{(5/4-\kappa)\wedge 1/2} \right) = \mathcal{O}_P \left( h^{1/4} \right),
$$

(A.15)

where $(Z'_2 M Z_2)^{-1} = \mathcal{O}_P \left( 1 \right)$ and the term $\mathcal{O}_P(h^{(5/4-\kappa)\wedge 1/2})$ term follows from equation (S.33) in CP (2020a). So the right-hand side of (A.15) is less than $\varepsilon/4$ in probability for large $T$. The second term of (A.12) can be dealt with as in equation (S.41) in CP (2020a). We deduce that the second term of (A.12) is such that

$$
K^{-1} h^{-\left(1-\kappa\right)-1/2} \delta_h' Z'_\Delta M e = \frac{h^{-1/2}}{K h^{1-\kappa}} \frac{\delta_h' \sum_{k=T_b+1}^{T_0^0} z_{kh} \tilde{e}_{kh}}{h^{1-\kappa}} - \frac{h^{-1/2}}{h^{1-\kappa}} \delta_h' \left( \sum_{k=T_b+1}^{T_0^0} z_{kh} x_{kh}' \right) (X'X)^{-1} (X'e) \\
\leq h^{-1/2} \frac{1}{K h^{1-\kappa}} \frac{\delta_h' \sum_{k=T_b+1}^{T_0^0} z_{kh} \tilde{e}_{kh} - C}{K} h^{-1/4} \left( \delta^0 \right) \left( \sum_{k=T_b+1}^{T_0^0} z_{kh} x_{kh}' \right) (X'X)^{-1} (X'e) \\
\leq h^{-1/2} \frac{1}{K h^{1-\kappa}} \frac{\delta_h' \sum_{k=T_b+1}^{T_0^0} z_{kh} \tilde{e}_{kh} - h^{-1/4} \mathcal{O}_P \left( 1 \right) \mathcal{O}_P \left( h^{1/2} \right)}{h^{-1/4}}.
$$

(A.16)

Thus, using (A.12), (A.5) can be written as

$$
\frac{h^{-1/2}}{K h^{1-\kappa}} 2 \delta_h' Z'_0 M e \pm \frac{h^{-1/2}}{K h^{1-\kappa}} 2 \delta_h' Z'_\Delta M e \pm \frac{h^{-1/2}}{K h^{1-\kappa}} 2 \delta_h' \left( Z'_\Delta M Z_2 \right) (Z'_2 M Z_2)^{-1} Z_2 M e - 2 \frac{h^{-1/2}}{K h^{1-\kappa}} \delta_h' \left( Z'_0 M e \right) \\
\leq \frac{h^{-1/2}}{K h^{1-\kappa}} \left( \delta^0 \right)' \sum_{k=T_b+1}^{T_0^0} z_{kh} \tilde{e}_{kh} \pm \mathcal{O}_P \left( h^{1/4} \right),
$$

in view of (A.15) and (A.16). It remains to consider (A.6). We can use the results in CP (2020a). In particular, following the steps from their equation (S.42)-(S.43) to their displayed equation before (S.44), yields that (A.6) is stochastically small uniformly in $T_b \in T_{K, \eta}$ as $h \downarrow 0$. Combining all the results the
relationship in (A.11) holds, concludes the proof. □

A.2 Proofs of the Results of Section 4

The expression for $G_h, \overline{Q}_b$ and $Q_{\alpha,h}$ are given by, respectively, $G_h \left( N_b \right) = S_{e,h} \left( \delta^0, N_b \right), \overline{Q}_b \left( N_b \right) = D_h \left( N_b \right) + S_{e,h} \left( N_b \right)$ and $Q_{\alpha,h} \left( N_b \right) = D_h \left( N_b \right)$.

A.2.1 Proof of Lemma 4.1

Proof. We show that for any $\eta > 0$, there exists a $C_2 > 0$ such that

$$Q_h \left( \delta_h, N_b \right) - Q_h \left( \delta_h, N_b^0 \right) < -C_2,$$

for every $N_b$ that satisfies $\left| N_b - N_b^0 \right| \geq \eta$. Recall the decomposition in (A.8), then (A.17) can be proved by showing that for any $\epsilon > 0$,

$$P \left( \sup_{\left| N_b - N_b^0 \right| \geq \eta} S_{e,h} \left( \delta_h, N_b \right) \geq \inf_{\left| N_b - N_b^0 \right| \geq \eta} \left| T_b - T_b^0 \right| D_h \left( \delta_h, N_b \right) \right) < \epsilon. \quad (A.18)$$

Suppose that $N_b < N_b^0$. The case with $N_b \geq N_b^0$ can be proved similarly. By definition, note that $\left| N_b - N_b^0 \right| \geq \eta$ is equivalent to $\left| T_b - T_b^0 \right| \geq T \eta$. Then,

$$P \left( \sup_{\left| N_b - N_b^0 \right| \geq \eta} h^{-1/2} \left| S_{e,h} \left( \delta_h, N_b \right) \right| \geq \inf_{\left| N_b - N_b^0 \right| \geq \eta} h^{-1/2} \left| T_b - T_b^0 \right| D_h \left( \delta_h, N_b \right) \right)$$

$$\leq P \left( D_{\eta,h}^{-1} \sup_{\left| T_b - T_b^0 \right| \geq T \eta} h^{-1/2} \left| S_{e,h} \left( \delta_h, N_b \right) \right| \geq \eta \right),$$

where $D_{\eta,h} = T \inf_{\left| T_b - T_b^0 \right| \geq T \eta} h^{-1/2} \overline{D}_h \left( N_b \right)$. Lemma A.8 below shows that $D_{\eta,h}$ is positive and bounded away from zero. Thus, it is sufficient to show that $\sup_{\left| T_b - T_b^0 \right| \geq T \eta} h^{-1/2} \left| S_{e,h} \left( \delta_h, N_b \right) \right| = o_P \left( 1 \right)$. The latter result was shown to hold by CP (2020a) [cf. (S.31)]. Thus,

$$P \left( \overline{D}_{\eta,h}^{-1} \sup_{\left| T_b - T_b^0 \right| \geq T \eta} h^{-1/2} \left| S_{e,h} \left( \delta_h, N_b \right) \right| \geq \eta \right) < \epsilon,$$

which implies (A.18) and concludes the proof. □

Lemma A.8. For any $B > 0$, let $D_{B,h} = \inf_{\left| T_b - T_b^0 \right| > TB} h^{-1/2} \overline{D}_h \left( \delta_h, N_b \right)$. There exists an $A > 0$ such that for every $\epsilon > 0$, there exists a $B < \infty$ such that $P \left( D_{B,h} \geq A \right) \leq 1 - \epsilon$. That is, $D_{B,h}$ is positive and bounded away from zero with high probability.

Proof. It follows from Lemma S.A.8 in CP (2020a). □

A.2.2 Proof of Theorem 4.1

We shall first prove a number of preliminary lemmas. We need the following additional notation. Recall that $l \in L$ and thus there exist some real number $B$ sufficiently large and some $a$ sufficiently small such
Lemma A.10. \( \inf_{|u| > B} l(u) - \sup_{|u| \leq B} l(u) \geq 0. \) (A.19)

Let \( \zeta_h(u) = \exp(\gamma_h \bar{G}_h(u) - A^0(u)), \Gamma_M = \{ u \in \mathbb{R} : M \leq |u| < M + 1 \} \cap \Gamma_h \) and set

\[
J_{1, M} \triangleq \int_{\Gamma_M} \zeta_h(u) \pi_h(u) \, du, \quad J_2 \triangleq \int_{\Gamma_h} \zeta_h(u) \pi_h(u) \, du. \quad (A.20)
\]

The proof involves showing the weak convergence toward the Gaussian process \( \mathcal{V}(u) = \mathcal{W}(u) - A^0(u) \) where

\[
\mathcal{W}(u) = \begin{cases} 2(\delta^0)' \mathcal{W}_1(u), & u \leq 0 \\ 2(\delta^0)' \mathcal{W}_2(u), & u > 0, \end{cases} \quad A^0(u) = \begin{cases} |u|(\delta^0)' \Sigma_{Z, 1} \delta^0, & u \leq 0 \\ u(\delta^0)' \Sigma_{Z, 2} \delta^0, & u > 0. \end{cases}
\]

Finally, we introduce the following class of functions. The function \( f_h : \mathbb{R} \to \mathbb{R} \) is said to belong to the family \( F \) if it possesses the following properties: (1) For a fixed \( h, f_h(x) \uparrow \infty \) monotonically as \( x \uparrow \infty \) and \( x \in [0, \infty) \); (2) For any \( b < \infty, x^b \exp(-f_h(x)) \to 0 \) as \( h \downarrow 0 \), and \( x \to \infty \).

For \( v < 0 \) define \( Q^0_{0, h}(v) \triangleq - (\delta^0)'(\sum_{k=T^0_0}^{T^0_k} z_{kh} z_{kh}') \delta^0_0, \) and \( G^*_v(v) \triangleq 2(\delta^0)'(\sum_{k=T^0_0}^{T^0_k} z_{kh} \bar{e}_{kh}), \) where \( \bar{e}_{kh} \sim \mathcal{N}(0, \sigma^2_e(k-1)h). \) Define analogously \( Q^*_{0, h}(v) \) and \( G^*_v(v) \) for \( v > 0 \), so that \( Q^*_h(v) = \bar{Q}_{0, h}(v) + G^*_h(v) \). The following lemma follows from Proposition 3.2.

**Lemma A.9.** Let \( v \in \Gamma^*. \) If \( v < 0 \), then \( Q^*_0, h(v) \overset{u.c.p.}{\to} |Z, Z|_1 \) where \( |Z, Z|_1 \triangleq [Z, Z]_{h_{[N^0_0/h]} - [Z, Z]_{h_{[t_e/h]}}, t_e} \triangleq N^0_0 + v. \) If \( v > 0 \), then \( Q^*_0, h(v) \overset{u.c.p.}{\to} [Z, Z]_2 \) where \( [Z, Z]_2 \triangleq [Z, Z]_{h_{[t_e/h]} - [Z, Z]_{h_{[N^0_0/h]}}, t_e} \triangleq N^0_0 + v. \)

**Lemma A.10.** We have \( h^{-1/2}G_v^*(v) \Rightarrow \mathcal{W}(v) \) in \( \mathbb{D}_b(C) \), where \( C \subset \mathbb{R} \) is a compact set and \( \mathcal{W}(v) = ((\delta^0)'\Omega_1 \delta^0)^{1/2} W_1(-v) \) if \( v < 0 \) and \( \mathcal{W}(v) = ((\delta^0)'\Omega_2 \delta^0)^{1/2} W_2(v) \) if \( v > 0. \)

**Proof.** This result also follows from Proposition 3.2 adapted to the case of stationary regimes. Hence, \( h^{-1/2}G_v^*(v) \Rightarrow \mathcal{W}(v) \) in \( \mathbb{D}_b(C). \) \( \square \)

**Lemma A.11.** On \( \{ u \leq K \}, \gamma \bar{Q}_h(u) \overset{d}{=} \bar{Q}_h(u) \Rightarrow \mathcal{V}(v), \) where \( \overset{d}{=} \) signifies equivalence in distribution and \( v \in \Gamma^* = (-\partial N^0_0, \partial (N - N^0_0)). \) On the other hand, \( \exp(\mathcal{V}(v)) \overset{P}{\to} 0 \) for all \( v \notin \Gamma^*. \)

**Proof.** Assume \( u \leq 0 \) (i.e., \( N_b \leq N^0_0 \)). We now employ a change of time scale as in CP (2020a). On the old time scale, given \( \{ |u| \leq K \}, N_b(u) \) varies on the time interval \( [N^0_0 - Kh^{1-\kappa}, N^0_0 + Kh^{1-\kappa}] \) for some \( K < \infty. \) Lemma A.6 shows that the asymptotic behavior of \( Q_T(T_b(u)) - Q_T(T^0_b) \) is determined by \( -\delta^0_h(Z'\Delta \Delta) \delta^0_h \pm 2\delta^0 \bar{c}_h(Z'\Delta e). \) Next, observe that scaling the criterion function \( Q_T(T_b(u)) - Q_T(T^0_b) \) by \( \psi_{h, 1} \) has the effect of changing the time scale \( s \mapsto t \triangleq \psi_{h, 1}s. \) That is, recall the processes \( Z_s \) and \( \bar{e}_s \) as defined in (2.3) and (3.1), respectively, and let \( Z_{\psi, s} \triangleq \psi_{h, 1}^{-1/2} Z_s, W_{\psi, e, s} \triangleq \psi_{h, 1}^{-1/2} W_{e, s}. \) Then, for \( s \in [N^0_0 - Kh^{1-\kappa}, N^0_0 + Kh^{1-\kappa}], \)

\[
Z_{\psi, t} = \psi_{h, 1}^{-1/2} \sigma_{Z, s} dW_{Z, t}, \quad W_{\psi, e, s} = \psi_{h, 1}^{-1/2} \sigma_{e, s} dW_{e, s}. \quad (A.21)
\]

With \( s \in [N^0_0 - Kh^{1-\kappa}, N^0_0 + Kh^{1-\kappa}] \) and \( v = \psi_{h, 1}^{-1} (N_b^0 - s) \), using the properties of \( W_{e, s} \) and the \( \mathcal{F}_{s^{*}} \) measurability of \( \sigma_{Z, s}, \sigma_{e, s}, \) we have

\[
Z_{\psi, t} = \sigma_{Z, t} dW_{Z, t}, \quad W_{\psi, e, t} = \sigma_{e, t} dW_{e, t}, \quad t \in \left[-\partial N^0_0, \partial \left(N - N^0_0\right)\right]. \quad (A.22)
\]

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This results in a change of time scale which we apply to $Z'_\Delta Z/\psi_h$ and $Z'_\Delta \tilde{e}/\psi_h$ so that

$$Z'_\Delta Z/\psi_h = \sum_{k=T_0^h+|v/h]}^{T_0^h} z_{\psi, kh} z'_\psi, \quad Z'_\Delta \tilde{e}/\psi_h = \sum_{k=T_0^h+|v/h]}^{T_0^h} z_{\psi, kh} \tilde{e}_{\psi, kh}, \quad (A.23)$$

where $z_{\psi, kh} = z_{kh}/\sqrt{\psi_h}$ and $\tilde{e}_{\psi, kh} = \tilde{e}_{kh}/\sqrt{\psi_h}$ with $v \in \Gamma^*$. In view of Condition 1, $\gamma_h \asymp h^{-1/2} \psi_h^{-1}$. By Lemma A.6 and (A.23),

$$\left( Q_h (N_b) - Q_h \left( N_0^0 \right) \right)/\psi_h = -\delta_h (Z'_\Delta Z_\Delta) \delta_h \pm 2\delta_h (Z'_\Delta \tilde{e}) + o_\psi (h^{1/2}).$$

or $\gamma_h Q_h (N_b) = - (\delta_0)' Z'_\Delta Z_\Delta (\delta_0) + 2 (\delta_0)' (h^{-1/2} Z'_\Delta \tilde{e}) + o_\psi (1)$ where $\epsilon_k \sim \text{i.n.d.} \mathcal{N} (0, \sigma^2_{h,k})$, $\sigma_{h,k} = O(h^{-1/4}) \sigma_{e,k}$, and $\tilde{e}_k$ is the normalized error [i.e., $\tilde{e}_k \sim \text{i.n.d.} \mathcal{N}(0, \sigma^2_{e,k})$] from (3.1). In summary, $\gamma_h Q_h (u) \overset{d}{=} \tilde{Q}_h^*(v)$, while by Lemma A.9-A.10, $\tilde{Q}_h^*(v) \Rightarrow \mathcal{Y} (v)$ on $\Gamma^*$. For $|u| > K$ and $K$ large enough, we have $\gamma_h Q_h (v) \downarrow -\infty$ in probability upon using the same arguments as in Lemma A.12-A.13 and thus we use the shorthand notation $\exp (\mathcal{Y} (u)) \overset{P}{\rightarrow} 0$ for all $v \not\in \Gamma^*$. □

**Lemma A.12.** For any $l \in \mathcal{L}$ and any $\epsilon > 0$,

$$\liminf_{h \downarrow 0} \mathbb{P} \left[ \int_{|u| \leq \eta T^{1-k}} l (s-u) \exp (\gamma_h \tilde{Q}_h (u)) \pi (N_0^0 + u/r_h) < \epsilon \right] \geq 1 - \epsilon, \quad (A.24)$$

for sufficiently large $K$ and small $\eta > 0$.

**Proof.** Since $l \in \mathcal{L}$, we have $l (s-u) \leq 1 + |s-u|^m$ for $m > 0$. Thus, we shall find a sufficiently large $K$ such that

$$\exp (\gamma_h \tilde{Q}_h (u)) \pi_h (N_0^0 + u/r_h) \overset{P}{\rightarrow} 0, \quad K \leq |u| \leq \eta T^{1-k}, \quad (A.25)$$

as $h \downarrow 0$. By Assumption 4.3, $\pi (\cdot)$ satisfies $\pi (N_0^0) < C_\pi$, for some real number $C_\pi < \infty$. Thus, $\pi (\cdot)$ will play no role in proving (A.25). By Lemma A.7, for large $K$ there exists a $C > 0$ such that

$$\liminf_{h \downarrow 0} \mathbb{P} \left[ \sup_{K \leq |u| \leq \eta T^{1-k}} Q_h (\delta_h, N_b) - Q_h (\delta_h, N_0^0) < -C \right] = 1.$$

Consequently, $\gamma_h \tilde{Q}_h (u)$ diverges to $-\infty$ for values of $u$ satisfying $K \leq |u| \leq \eta T^{1-k}$. By the property of the exponential function, we have for some finite $C_1$, $C_2 > 0$, $\exp (\gamma_h \tilde{Q}_h (u)) = C_1 \exp (-\kappa \cdot T^{3/2-\kappa} C_2) \overset{P}{\rightarrow} 0$ for all $K \leq |u| \leq \eta T^{1-k}$. Combining all the arguments above, the latter result implies (A.24), which concludes the proof. □

**Lemma A.13.** For every $l \in \mathcal{L}$, and any $\epsilon, \eta > 0$, we have

$$\liminf_{h \downarrow 0} \mathbb{P} \left[ \int_{|u| \leq \eta T^{1-k}} l (s-u) \exp (\gamma_h \tilde{Q}_h (u)) \pi_h (N_0^0 + u/r_h) < \epsilon \right] \geq 1 - \epsilon. \quad (A.26)$$

**Proof.** Since $l \in \mathcal{L}$, we have $l (s-u) \leq 1 + |s-u|^m$ for $m > 0$. We shall show

$$\int_{|u| \geq \eta T^{1-k}} (1 + |s-u|^m) \exp (\gamma_h \tilde{Q}_h (u)) \pi_h (N_0^0 + u/r_h) \, du = o_\psi (1). \quad (A.27)$$
Lemma A.14. \textbf{Proof.} The proof is given only for the case \( i = 1 \) where \( u \) and some constants \( c_r \) which depend on \( r \) only, we have

\[
\mathbb{E} \left[ \left( \zeta_h^{1/2r} (u_2) - \zeta_h^{1/2r} (u_1) \right)^{2r} \right] \leq c_r \left[ \left( \frac{\partial^0}{u_2 - u_1 | \Sigma_i} \frac{\delta^0}{\delta^0} \right) \left( |u_2 - u_1| | \Sigma_i | \delta^0 \right) \right] \leq c_r |u_2 - u_1|^r,
\]

where \( i = 1 \) if \( u_1 < 0 \) and \( i = 2 \) if \( u_1 > 0 \). We follow Lemma III.5.2 in Ibragimov and Has'minskii (1981). Let \( \mathcal{V} (u_i) = \exp (\mathcal{V} (u_i)) \) for \( i = 1, 2 \) where \( \mathcal{V} (u_i) = \mathcal{V} (u_i) - A^0 (u_i) \) and \( \mathcal{V}_{u_1} (u_2) \triangleq \exp (\mathcal{V} (u_2) - \mathcal{V} (u_1)) \). We have

\[
\mathbb{E}_{u_1} \left[ \mathcal{V}_{u_1}^{r/2r} (u_2) \right] = \exp \left( \frac{1}{2} \left( \frac{j}{2r} \right)^2 \left( \frac{\partial^0}{u_2 - u_1 | \Sigma_i} \frac{\delta^0}{\delta^0} \right) \left( |u_2 - u_1| | \Sigma_i | \delta^0 - \frac{j}{2r} |A^0 (u_2) - A^0 (u_1)| \right) \right). \tag{A.28}
\]

Letting \( d \triangleq \exp \left( j (2r)^{-1} \left( \frac{\delta^0}{u_2 - u_1 | \Sigma_i} \frac{\delta^0}{\delta^0} - |A^0 (u_2) - A^0 (u_1)| \right) \right) \), we have

\[
\mathbb{E} \left[ \left( \mathcal{V}^{1/2r} (u_2) - \mathcal{V}^{1/2r} (u_1) \right)^{2m} \right] = \frac{2r}{j} \left( \frac{2r}{j} \right)^j (-1)^j d^{j/2r}.
\]

We need to study three different cases. Let \( \varpi \triangleq 2 \left( \frac{\delta^0}{u_2 - u_1 | \Sigma_i} \frac{\delta^0}{\delta^0} - |A^0 (u_2) - A^0 (u_1)| \right) \).

(1) \( \varpi < 0 \). Note that

\[
d = \exp \left( \frac{j}{r} \left( \frac{\delta^0}{u_2 - u_1 | \Sigma_i} \frac{\delta^0}{\delta^0} - 2 \left( \frac{\delta^0}{u_2 - u_1 | \Sigma_i} \frac{\delta^0}{\delta^0} \right) + \varpi \right) \right) = \exp \left( -\frac{2r - j}{r} \left( \frac{\delta^0}{u_2 - u_1 | \Sigma_i} \frac{\delta^0}{\delta^0} \right) e^{\varpi} \right).
\]
which then yields
\[
\mathbb{E} \left[ \left( \mathcal{V}^{1/2r} (u_2) - \mathcal{V}^{1/2r} (u_1) \right)^{2r} \right] \leq p_r (a),
\]  
(A.29)

where \( p_r (a) \) is a polynomial of degree \( r^2 - r \), and the last inequality follows from \( 1 - e^{-c} \leq c, \) for \( c > 0 \). Now, write \( \xi_h^{1/2r} (u_2, u_1) = \mathcal{V}^{1/2r} (u_2) - \mathcal{V}^{1/2r} (u_1) \). By Lemma A.10, the continuous mapping theorem and (A.30), we have \( \lim_{h \to 0} \mathbb{E} [\xi_h^{1/2r} (u_2, u_1)]^2 \leq (1 - a)^r \overline{p}_r (a) \). Since \( j \leq 2r \), we set \( c_r = \max_{0 \leq a \leq 1} \overline{p}_r (a) / r^r < \infty \) and the claim of the lemma follows since \( \| \delta^0 \|, \Sigma_i \leq \infty \). \( \Box \)

**Lemma A.15.** Let \( 0 < C < \infty \) and \( u_1, u_2 \in \mathbb{R} \) being of the same sign satisfying \( 0 < |u_1| < |u_2| < K < \infty \). Then, for all \( h \) sufficiently small, we have
\[
\mathbb{E} \left[ \left( \mathcal{V}^{1/4} (u_2) - \mathcal{V}^{1/4} (u_1) \right)^4 \right] \leq C_2 |u_2 - u_1|^2.
\]  
(A.31)

Furthermore, for some constant \( C_1 \) as in Lemma A.14 we have
\[
\mathbb{P} [\xi_h (u) > \exp (-3C_1 |u| / 2)] \leq \exp (-C_1 |u| / 4).
\]  
(A.32)

**Proof.** Use Lemma A.14 with \( r = 2 \) to verify (A.31). For (A.32), assume \( u > 0 \). By Markov’s inequality and Lemma A.14,
\[
\mathbb{P} [\xi_h (u) > \exp (-3C_1 |u| / 2)] \leq \exp (3C_1 |u| / 4) \mathbb{E} \left[ \mathcal{V}^{1/2} (u) \right] \leq \exp \left( 3C_1 |u| / 4 \right). \Box
\]

**Lemma A.16.** Under the conditions of Lemma A.15, for any \( \omega > 0 \) there exists a finite real number \( c_\omega \) and a \( \overline{h} \) such that for all \( h < \overline{h} \), \( \mathbb{P} [\sup_{|u| > M} \xi_h (u) > M^{-\infty}] \leq c_\omega M^{-\infty} \).
Proof. It can be easily shown to follow from the previous lemma. □

Lemma A.17. For a small \( \epsilon \leq \tau \), where \( \tau \) may depend on \( \pi (\cdot) \), there exists \( 0 < C < \infty \) such that

\[
\mathbb{P} \left[ \int_0^\epsilon \zeta_h (u) \pi_h (u) \, du < \epsilon \pi^0 \right] < C \epsilon^{1/2}. \tag{A.33}
\]

Proof. Since \( \mathbb{E} (\zeta_h (0)) = 1 \) while \( \mathbb{E} (\zeta_h (u)) \leq 1 \) for sufficiently small \( h > 0 \),

\[
\mathbb{E} |\zeta_h (u) - \zeta_h (0)| \leq \left( \mathbb{E} \left( \zeta_h^{1/2} (u) + \zeta_h^{1/2} (0) \right)^2 \mathbb{E} \left( \left( \zeta_h^{1/2} (u) - \zeta_h^{1/2} (0) \right)^2 \right) \right)^{1/2} \leq C |u|^{1/2}, \tag{A.34}
\]

where we used Lemma A.14 with \( r = 1 \). By Assumption 4.3, \( |\pi_h (u) - \pi^0| \leq B (T^{1-\kappa})^{-1} |u| \), with \( B > 0 \). Given that \( |u| < \tau \), it holds that \( \pi^0 / 2 < \pi_h (u) \). Thus, for a small \( \epsilon > 0 \), \( \int_0^\epsilon \zeta_h (u) \pi_h (u) \, du > 2^{-1} \pi^0 \int_0^\epsilon \zeta_h (u) \, du \). We can then use \( \zeta_h (0) = 1 \) to yield,

\[
\mathbb{P} \left[ \int_0^\epsilon \zeta_h (u) \pi_h (u) \, du < 2^{-1} \epsilon \right] \leq \mathbb{P} \left[ \int_0^\epsilon (\zeta_h (u) - \zeta_h (0)) \, du < -2^{-1} \epsilon \right] \leq \mathbb{P} \left[ \int_0^\epsilon |\zeta_h (u) - \zeta_h (0)| \, du > 2^{-1} \epsilon \right].
\]

By Markov’s inequality together with inequality (A.34), the right-hand side above is less than or equal to \( 2 \epsilon^{-1} \int_0^\epsilon \mathbb{E} |\zeta_h (u) - \zeta_h (0)| \, du < 2C \epsilon^{1/2} \). □

Lemma A.18. For \( f_h \in \mathcal{F} \), and \( M \) sufficiently large, there exist constants \( c, C > 0 \) such that

\[
\mathbb{P} \left[ J_{1,M} > \exp (-c f_h (M)) \right] \leq C \left( 1 + M^C \right) \exp (-c f_h (M)). \tag{A.35}
\]

Proof. Since \( u \in \Gamma_h \), and given the differentiability of \( \pi (\cdot) \), we consider for simplicity the case of the uniform prior (i.e., \( \pi_h (u) = 1 \) for all \( u \)). We divide the open interval \( \{ u : M \leq |u| < M + 1 \} \) into \( I \) disjoint segments and denote the \( i \)-th segment by \( \Pi_i \). Choose a point \( u_i \in \Pi_i \) and define \( J_{1,M}^\Pi \triangleq \sum_{i \in I} \zeta_h (u_i) \text{Leb} (\Pi_i) = \sum_{i \in I} \int_{\Pi_i} \zeta_h (u) \, du \). Then,

\[
\mathbb{P} \left[ J_{1,M}^\Pi > 4^{-1} \exp (-c f_T (M)) \right] \leq \mathbb{P} \left[ \max_{i \in I} \zeta_h^{1/2} (u_i) (\text{Leb} (\Gamma_M))^{1/2} > (1/2) \exp (-f_h (M) /2) \right]
\]

\[
\leq \sum_{i \in I} \mathbb{P} \left[ \zeta_h^{1/2} (u_i) > (1/2) (\text{Leb} (\Gamma_M))^{-1/2} \exp (-f_h (M) /2) \right]
\]

\[
\leq 2 I (\text{Leb} (\Gamma_M))^{1/2} \exp (-f_h (M) /12), \tag{A.36}
\]

where the last inequality follows from applying the second part of Lemma A.15 to each summand. For a sufficiently large \( M \), \( \exp (-f_h (M) /12) < 1/2 \) and thus,

\[
\mathbb{P} \left[ J_{1,M} > \exp (-f_h (M) /2) \right] \leq \mathbb{P} \left[ J_{1,M} - J_{1,M}^\Pi > 2^{-1} \exp (-f_h (M) /2) \right] + \mathbb{P} \left[ J_{1,M}^\Pi > \exp (-f_h (M)) \right]. \tag{A.37}
\]

Let us focus on the first term:

\[
\mathbb{E} \left[ J_{1,M} - J_{1,M}^\Pi \right] \leq \sum_{i \in I} \int_{\Pi_i} \mathbb{E} \left| \zeta_h^{1/2} (u) - \zeta_h^{1/2} (u_i) \right| \, du
\]

\[
\leq \sum_{i \in I} \int_{\Pi_i} \left( \mathbb{E} \left| \zeta_h^{1/2} (u) + \zeta_h^{1/2} (u_i) \right| \mathbb{E} \left| \zeta_h^{1/2} (u) - \zeta_h^{1/2} (u_i) \right| \right)^{1/2} \, du
\]

\[
\leq C (1 + M)^C \sum_{i \in I} \int_{\Pi_i} |u_i - u|^{1/2} \, du,
\]
where the last inequality uses Lemma A.14 and the fact that \( u < M + 1 \). Note that each summand on the right-hand side above is less than \( C (M^{-1})^{3/2} \). Thus, we can find numbers \( C_1 \) and \( C_2 \) such that

\[
\mathbb{E} \left[ J_{1,M} - J_{1,M}^H \right] \leq C_1 \left( 1 + M^{C_2} \right) I^{-1/2}.
\] (A.38)

Using (A.36) and (A.38) into (A.37), we have

\[
\mathbb{P} \left[ J_{1,M} > \exp (-f_h(M)/2) \right] \leq C_1 \left( 1 + M^{C_2} \right) I^{-1/2} + 2I \left( \text{Leb}(\Gamma_M) \right)^{1/2} \exp (-f_h(M)/12).
\]

The claim of the lemma follows choosing a \( I \) that satisfies \( 1 \leq I^{3/2} \exp (-f_h(M)/12) \leq 2 \). \( \Box \)

**Lemma A.19.** For \( f_h \in F \), and \( M \) sufficiently large, there exist constants \( c, C > 0 \) such that

\[
\mathbb{E} \left[ J_{1,M}/J_2 \right] \leq C \left( 1 + M^C \right) \exp (-c f_h(M)).
\] (A.39)

**Proof.** Note that \( J_{1,M}/J_2 \leq 1 \) and thus, for an arbitrary \( \epsilon > 0 \),

\[
\mathbb{E} \left[ J_{1,M}/J_2 \right] \leq \mathbb{P} \left[ J_{1,M} > \exp (-c f_h(M)/2) \right] + 4\epsilon^{-1} \exp (-c f_h(M)) + \mathbb{P} \left[ \int_{\Gamma_h} \zeta_h(u) du < \epsilon/4 \right]
\]

where we used Lemma A.18 for the first term. In view of the relationship in (A.33), \( \mathbb{P} \left[ \int_{\Gamma_h} \zeta_h(u) du < \epsilon/4 \right] \leq C \epsilon^{1/2} \). To conclude the proof, choose \( \epsilon = \exp ((-2c/3) f_h(M)) \). \( \Box \)

**Proof of Theorem 4.1.** Let \( p_{1,h}(u) \triangleq \tilde{p}_h(u)/\pi_h \), where \( \tilde{p}_h(u) = \exp(\gamma_h(\tilde{G}_h(u) + \tilde{Q}_{0,h}(u))) \) and \( \pi_h \triangleq \int_{\Gamma_h} p_{1,h}(w) dw \). By definition, \( \tilde{N}_{GL} = \tilde{N}_{GL} \) minimizes

\[
\int_{\Gamma^0} l(r_h(s-u)) \exp \left( \gamma_h \left( \tilde{G}_h(u) + \tilde{Q}_{0,h}(u) \right) \right) \pi(u) du, \quad s \in \Gamma^0.
\]

Changing variables and using simple manipulations, the expression above is equal to

\[
r_h^{-1} \tilde{p}_h \int_{\Gamma_h} l \left( r_h \left( s - N_b^0 \right) - u \right) p_{1,h}(u) \pi_h(u) du,
\] (A.40)

from which it follows that \( \xi_{t,h} \triangleq \vartheta^{-1} T_1^{-1-\kappa} (\tilde{N}_{GL} - \lambda_0) \) is the minimum of the function

\[
\Psi_{t,h} \left( s \right) \triangleq \int_{\Gamma_h} l(s-u) \frac{\tilde{p}_h(u) \pi \left( N_b^0 + u/r_h \right)}{\int_{\Gamma_h} \tilde{p}_h(w) \pi \left( N_b^0 + w/r_h \right) dw} du.
\] (A.41)

Lemma A.11 shows that, under Condition 1, the normalizing factor \( \gamma_h \) acts as a change of time scale \( s \mapsto \psi_h^{-1} s \) since \( r_h \sim \psi_h^{-1} \), where \( a_h \sim b_h \) signifies \( b_h/c \leq a_h \leq c b_h \) for some constant \( c \). The change of time scale then implies that the sample criterion \( \tilde{Q}_h(N_b) \) is evaluated at \( N_b = N_b^0 + \vartheta v \). It also suggests the following change of variable through the substitution \( a = \vartheta \psi_h u \),

\[
\Psi_{t,h} \left( s \right) = \int_{\Gamma^*} l \left( (\vartheta \psi_h)^{-1} (\vartheta \psi_h s - a) \right) \frac{\exp \left( \tilde{G}_h^* (a) + \tilde{Q}_{0,h}^* (a) \right) \pi \left( N_b^0 + a \right)}{\int_{\Gamma^*} \exp \left( \tilde{G}_h^* (w) + \tilde{Q}_{0,h}^* (w) \right) \pi \left( N_b^0 + w \right) dw} da,
\] (A.42)

where \( \Gamma^* = (-\vartheta N_b^0, \vartheta (N - N_b^0)) \) and the Quasi-prior is defined on the “fast time scale”. This implies that \( \Psi_{t,h} \left( s \right) \triangleq \Psi_{t,h} \left( \vartheta \psi_h s \right) \) is minimized by \( \xi_{t,h} \triangleq T_h(\tilde{N}_{GL} - \lambda_0) \). The next step involves showing the finite-dimensional convergence in distribution of the function \( \Psi_{t,h} \left( s \right) \) to \( \Psi^* \left( s \right) \) and the tightness of the sequence of functions \( \left( \Psi_{t,h} \left( s \right) \right) \) on the space \( \mathbb{C}_b(\Gamma^*) \). This gives the weak convergence \( \Psi_{t,h} \Rightarrow \Psi^* \) on
The final part of the proof requires showing that (a) any element of the set of minimizers of the sample criterion is stochastically bounded, and (b) the length of such set converges to zero as $h \downarrow 0$. Note that we can deduce certain tail properties of $\Psi_{l,h}^*$ from that of $\Psi_{l,h}$ on $\{u \in \mathbb{R}, |u| \leq K\}$ as $K \to \infty$. Thus, we shall sometimes study the behavior of $\Psi_{l,h}$ below. Given the boundedness of $\pi(\cdot)$ [cf. Assumption 4.3], Lemma A.18-A.19 imply that, for any $\nu > 0$, we can find a $\tilde{h}$ such that for all $h < \tilde{h}$,

$$
E \left[ \int_{\Gamma_M} \frac{\exp \left( \gamma_h \tilde{G}_h (u) + \tilde{Q}_{0,h} (u) \right)}{\int_{\Gamma_h} \exp \left( \gamma_h G_h (u) + Q_{0,h} (u) \right) du} dw \right] \leq \frac{c_{\nu}}{M^\nu}.
$$

(A.43)

Furthermore, by Lemma A.12-A.13, for $K < \infty$,

$$
\Psi_{l,h} (s) = \int_{\{u \in \mathbb{R}, |u| \leq K\}} l (s - u) \frac{\bar{p}_h (u)}{\int_{\{u \in \mathbb{R}, |u| \leq K\}} \bar{p}_h (w) dw} du + o_P (1).
$$

The last two relationships extend to $\Psi_{l,h}^*$ (s) and in particular (A.43) shows the tail behavior of $\Psi_{l,h}$ (s). On the region $\{u \leq K\}$, the change of time scale $u \mapsto \psi_h u = v$ implies that we need to analyze the behavior of $\Psi_{l,h}^*$ (s). Thus, we show the convergence of the marginal distributions of the function $\Psi_{l,h}^*$ (s) to the marginals of $\Psi_l^*$ (s) defined in Definition 4.1. Choose a finite integer $n$, and arbitrary real numbers $a_j \ (j = 0, \ldots, n)$. Let $\zeta^*_h (v) = \tilde{G}_h^* (v) + \tilde{Q}_{0,h}^* (v)$. The estimate

$$
\sum_{j=1}^n a_j \int_{|v| \leq K} l (s_j - v) \zeta^*_h (v) \pi (N_b^0 + u) \pi (N_b^0 + v) dv + a_0 \int_{|v| \leq K} \zeta^*_h (v) \pi (N_b^0 + v) dv
$$

(A.44)

converges in distribution to the random variable

$$
\sum_{j=1}^n a_j \int_{|v| \leq K} l (s_j - v) \exp (\mathcal{V} (v)) \pi (N_b^0 + v) dv + a_0 \int_{|v| \leq K} \exp (\mathcal{V} (v)) \pi (N_b^0 + v) dv,
$$

where $\mathcal{V} (v) = \mathcal{W} (v) - A^0 (v)$, since by Lemma A.14-A.15 we can apply Theorem I.A.22 in Ibragimov and Has’minskii (1981). We can then use the Cramer-Wold Theorem [cf. Theorem 29.4 in Billingsley (1995)] to establish the convergence in distribution of the vector

$$
\int_{|v| \leq K} l (s_1 - v) \zeta^*_h (v) \pi (N_b^0 + v) dv, \ldots, \int_{|v| \leq K} l (s_n - v) \zeta^*_h (v) \pi (N_b^0 + v) dv, \int_{|v| \leq K} \zeta^*_h (v) \pi (N_b^0 + v) dv,
$$

to the marginal distributions of the vector

$$
\int_{|v| \leq K} l (s_1 - v) \exp (\mathcal{V} (v)) \pi (N_b^0 + v) dv, \ldots, \int_{|v| \leq K} l (s_n - v) \exp (\mathcal{V} (v)) \pi (N_b^0 + v) dv,
$$

$$
\int_{|v| \leq K} \exp (\mathcal{V} (v)) \pi (N_b^0 + v) dv.
$$

Note that the above integrals are equal to zero if $v \notin \Gamma^*$, because in the latter result there is an intermediate step involving the change of time scale. That is, $v \notin \Gamma^*$ corresponds to tail behavior as described in (A.43) and the equation right below it. Then, for any $K_1, K_2 < \infty$, the marginal distributions of

$$
\frac{\int_{|v| \leq K_1} l (s - v) \exp \left( \tilde{G}_h^* (v) + \tilde{Q}_{0,h}^* (v) \right) \pi (N_b^0 + v) dv}{\int_{|v| \leq K_2} \exp \left( \tilde{G}_h^* (w) + \tilde{Q}_{0,h}^* (w) \right) \pi (N_b^0 + w) dw},
$$
By the same reasoning, the marginal distributions of
\[ \int_{|w| \leq K_2} \exp \left( G_h^* (v) + Q_{0,h}^* (v) \right) \, dv, \]
converge to the distribution of \( \int_{|w| \leq M + 1} \exp(\mathcal{Y}(v)\pi(N_b^0 + v)) / \int_{|w| \leq K_2} \exp(\mathcal{Y}(w)) \pi(N_b^0 + w) \, dw \, dv. \) In view of (A.43) which gives a bound on the tail behavior of the mean of the Quasi-posterior on the original time scale, we have
\[
\Psi^*_{l,h} (s) = \int_{\Gamma^*} l(s - v) \frac{\exp(\mathcal{Y}(v)) \pi(N_b^0 + v) \, dv}{\int_{\Gamma^*} \exp(\mathcal{Y}(w)) \pi(N_b^0 + w) \, dw} + o_P (1). \tag{A.45}
\]
This is sufficient for the convergence of the finite-dimensional distributions of \( \Psi^*_{l,h} (s) \) to those of \( \Psi^* (s) \).

We now turn to the tightness of the sequence of functions \( \{\Psi^*_{l,h}(s)\} \). We shall show that the probability distributions in \( C([-K, K]) \) (i.e., the space of continuous functions on \([-K, K]\)) generated by the contractions of the functions \( \Psi^*_{l,h}(s) \) on \( s \in [-K, K] \) are tight. For any \( l \in \mathbb{L} \), we have the inequality \( l(u) \leq 2^r (1 + |u|^2)^r \), for some \( r \). Let
\[
H_K (\pi) = \int_{\mathbb{R}} \sup_{|s| \leq K, |y| \leq \pi} |l(s + y - u) - l(s - u)| \left( 1 + |u|^2 \right)^{-r-1} \, du.
\]
We show that \( \lim_{\pi \downarrow 0} H_K (\pi) = 0 \). Note that for any \( c > 0 \) we can choose a \( M \) such that
\[
\int_{|u| > M} \sup_{|s| \leq K, |y| \leq \pi} |l(s + y - u) - l(s - u)| \left( 1 + |u|^2 \right)^{-r-1} \, du < c.
\]
We apply Lusin’s theorem [cf. Section 3.3 of Royden and Fitzpatrick (2010)]. Let \( \mathcal{L} \) denote the upper bound of \( l(\cdot) \) on the set \( \{u \in \mathbb{R} : |u| \leq K + 2M\} \). By the measurability of \( l(\cdot) \), we can find a continuous function \( J(u) \) in the interval \( \{u : |v| \leq K + 2M\} \) which equals \( l(u) \) except on a set whose measure does not exceed \( c(2\mathcal{L})^{-1} \). Denote the modulus of continuity of \( J(\cdot) \) by \( w_J(\pi) \). Without loss of generality assume \( |J(u)| \leq \mathcal{L} \) on \( \{u : |u| \leq K + 2M\} \). Then,
\[
\int_{|u| > M} \sup_{|s| \leq K, |y| \leq \pi} |l(s + y - u) - l(s - u)| \left( 1 + |u|^2 \right)^{-r-1} \, du
\leq w_J(\pi) \int_{\mathbb{R}} \left( 1 + |u|^2 \right)^{-r-1} \, du + 2\mathcal{L} \text{Leb} \{u \in \mathbb{R} : |u| \leq K + 2M, l \neq J\},
\]
and \( \mathcal{L} \leq C w_J(\pi) + c \) for some \( C < \infty \). Accordingly, \( H_K(\pi) \leq C w_J(\pi) + 2c \), with the property that \( c \) can be chosen arbitrary small and \( w_J(\pi) \to 0 \) as \( \pi \downarrow 0 \) (holding for each fixed \( c \)) by definition. Definition 4.1 implies that we can find a number \( C < \infty \) such that
\[
\mathbb{E} \left[ \sup_{|s| \leq K, |y| \leq \pi} \left| \Psi^*_{l,h}(s + y) - \Psi^*_{l,h}(s) \right| \right]
\leq \int_{\mathbb{R}} \sup_{|s| \leq K, |y| \leq \pi} |l(s + y - u) - l(s - u)| \mathbb{E} \left( \frac{\zeta_h(u) \pi_h(u)}{\int_{\Gamma_h} \zeta_h(w) \pi_h(w) \, dw} \right) \, du \leq C H_K(\pi).
\]
The tightness of \( \Psi^*_{l,h}(s) \) on \( s \in \Gamma^* \) is established by using Markov’s inequality together with the above bound. It remains to study the oscillations of the minimum points of the sample function \( \Psi^*_{l,h} \). Consider
an open bounded interval \( A \) satisfying \( \mathbb{P} [\xi^0_t \in b(A)] = 0 \), where \( b(A) \) denotes the boundary of the set \( A \). Define the functionals \( H_A (\Psi) = \inf_{s \in A} \Psi^*_t (s) \) and \( H_{A^c} (\Psi) = \inf_{s \in A^c} \Psi^*_t (s) \), where \( A^c \) is the set complementary to \( A \). Let \( M_h \) denote the set of absolute minimum points of the function \( \Psi^*_t (s) \). By the definition of \( \hat{\lambda}_b^{GL} \) we have that \( \lim \sup_{h \downarrow 0} \mathbb{E} [l(Th(\hat{\lambda}_b^{GL} - \lambda_0))] < \infty \). This implies that any minimum point \( \xi \in M_h \) of the sample criterion function is uniformly stochastically bounded, i.e.,

\[
\lim_{K \rightarrow \infty} \mathbb{P} \left[ M_h \not\subset \left\{ s : |s| \leq K \right\} \right] = 0. \tag{A.46}
\]

Next, note that

\[
\mathbb{P} [M_h \subset A] = \mathbb{P} [H_A (\Psi) < H_{A^c} (\Psi), M_h \subset \{ s : |s| \leq K \}] \\
\geq \mathbb{P} [H_A (\Psi) < H_{A^c} (\Psi)] - \mathbb{P} [M_h \not\subset \{ s : |s| \leq K \}].
\]

Furthermore, the relationships

\[
\lim \inf_{h \downarrow 0} \mathbb{P} [M_h \subset A] \geq \mathbb{P} [H_A (\Psi) < H_{A^c} (\Psi)] - \mathbb{P} [M_h \not\subset \{ s : |s| \leq K \}],
\]

and \( \lim \sup_{h \downarrow 0} \mathbb{P} [M_h \subset A] \leq \mathbb{P} [H_A (\Psi) < H_{A^c} (\Psi)] \) are valid. As for the minimum point of the population criterion \( \Psi^*_t (\cdot) \), we have \( \mathbb{P} [\xi^0_t \in A] \leq \mathbb{P} [H_A (\Psi) < H_{A^c} (\Psi)] \), and \( \mathbb{P} [\xi^0_t \in A] + \mathbb{P} [|\xi^0_t| > K] \geq \mathbb{P} [H_A (\Psi) < H_{A^c} (\Psi)] \). The uniqueness assumption on the population criterion (cf. Assumption 4.2) and (A.46) leads to \( \lim_{K \rightarrow \infty} \{ \sup_{h \leq K} \mathbb{P} [M_h \not\subset \{ s : |s| \leq K \}] + \mathbb{P} [|\xi^0_t| > K] \} = 0 \), for a small \( k > 0 \). Hence,

\[
\lim_{h \downarrow 0} \mathbb{P} [M_h \subset A] = \mathbb{P} [\xi^0_t \in A]. \tag{A.47}
\]

In the last step of the proof, we show that the length of the set \( M_h \) converges to zero in probability as \( h \downarrow 0 \). Let \( A_d \) denote an interval in \( \mathbb{R} \) centered at the origin and of length \( d < \infty \). In view of (A.47), \( \lim_{d \rightarrow \infty} \sup_{h \downarrow 0} \mathbb{P} [M_h \not\subset A_d] = 0 \). Fix any \( \epsilon > 0 \) and divide \( A_d \) into admissible subintervals whose lengths do not exceed \( \epsilon/2 \). We have,

\[
\mathbb{P} \left[ \sup_{s_i, s_j \in M_h} |s_i - s_j| > \epsilon \right] \leq \mathbb{P} [M_h \not\subset A_d] + (1 + 2d/\epsilon) \sup \mathbb{P} \left[ H_A \left( \Psi^*_t (h) \right) = H_{A^c} \left( \Psi^*_t (h) \right) \right],
\]

where \( 1 + 2d/\epsilon \) is an upper bound on the number of subintervals and the supremum is taken over all possible open bounded subintervals \( A \subset A_d \). Given \( \Psi^*_t \Rightarrow \Psi^*_t \), we have \( \mathbb{P} [H_A (\Psi^*_t (h)) = H_{A^c} (\Psi^*_t (h))] \rightarrow \mathbb{P} [H_A (\Psi) = H_{A^c} (\Psi)] \) as \( h \downarrow 0 \). Since \( \mathbb{P} [H_A (\Psi) = H_{A^c} (\Psi)] = 0 \) and \( d \) can be chosen large, we have \( \mathbb{P} [\sup_{s_i, s_j \in M_h} |s_i - s_j| > \epsilon] = o (1) \). Given that \( \epsilon > 0 \) can be chosen arbitrarily small we have that the distribution of \( \xi_{t,h} = Th(\hat{\lambda}_b^{GL} - \lambda_0) \) converges to the distribution of \( \xi^0_t \). \( \square \)

### A.3 Proofs of Section 6

#### A.3.1 Proof of Theorem 6.1

The next tree lemmas correspond to Lemma A.29-A.31 from Casini and Perron (2020c), respectively.

**Lemma A.20.** For \( \varpi > 3/4 \), we have \( \lim_{h \downarrow 0} \lim \sup_{|s| \rightarrow \infty} \left| \widehat{\nu}_h (s) / |s|^\varpi \right| = 0 \) \( \mathbb{P} \)-a.s.

**Lemma A.21.** \( \{ \widehat{\nu}_h \} \) converges weakly toward \( \nu \) on compact subsets of \( \mathbb{D}_h \).
Lemma A.22. Fix $0 < a < \infty$. For $l \in L$ and any positive sequence $\{a_h\}$ satisfying $a_h \overset{P}{\rightarrow} a$,

$$
\int_{\mathbb{R}} |l(r - s)| \exp \left( \hat{\Psi}_h(s) \right) \exp (-a_h |s|) \, ds \overset{d}{\rightarrow} \int_{\mathbb{R}} |l(r - s)| \exp \left( \left( \mathcal{W}(s) \right) \right) \exp (-a |s|) \, ds.
$$

Proof of Theorem 6.1. Let $C \subset \mathbb{R}$ be compact. Suppose $\gamma_h$ satisfies Condition 1. Then,

$$
\hat{\Psi}_{l,h}^* (s) = \int_{\hat{\Gamma}^*} l(s - v) \frac{\exp \left( \hat{\mathcal{V}}_h(v) \right) \pi \left( \hat{N}_b^{GL} + v \right) \Gamma^*}{\int_{\hat{\Gamma}^*} \exp \left( \hat{\mathcal{V}}_h(w) \right) \pi \left( \hat{N}_b^{GL} + w \right) \, dw} \, dv + o_P(1).
$$

Using Lemma A.21, $\hat{\mathcal{V}}_h \Rightarrow \mathcal{V}$ in $D_b(C)$. From Assumption 6.1-(i), $\hat{\Gamma}^*$ can be replaced by $\Gamma^*$ for small enough $h$ if we show that the integral as a function of $\hat{N}_b^{GL}$ is continuous. Note that the integrand is Riemann integrable and thus the integral considered as a map is continuous. Since $\mathcal{V}$ is $\mathbb{P}$-a.s. continuous, then the desired result follows by the continuity of the composition of continuous functions. Using the same steps as in the proof of Theorem 4.1 together with Lemma A.22 yields the result. □
References